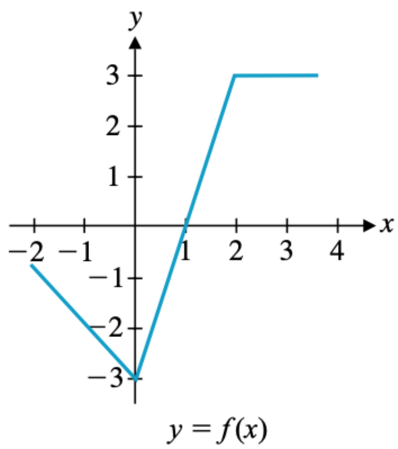
1. (38 points) Find the requested information.

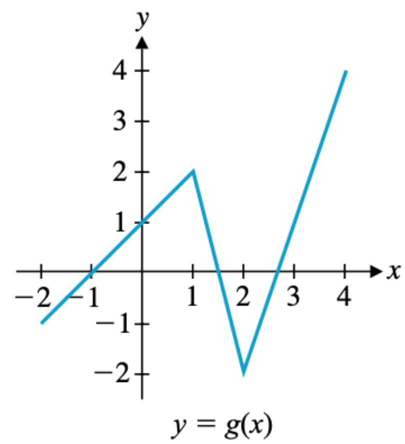
   (a) Find $dg/d\theta$ for $g(\theta) = \theta \sin(c^2\theta^2 + 3)$ and c a positive constant. Do not simplify your final answer.

   (b) Find df/dt for $f(t) = \dfrac{t^2 - 3t}{t^{2/3} + 5}$. Do not simplify your final answer.

   (c) Find the equation of the line tangent to $y^2 + \sin(xy) = 4$ at the point $(0, 2)$.

   (d) The graphs of $y = f(x)$ and $y = g(x)$ are given below.





   Find the derivative of $g(f(x))$, if it exists, at $(i) x = 0$, $(ii) x = 1$, and $(iii) x = 3$. If the derivative does not exist, state this and explain why it does not exist.

Solution:

(a) $\dfrac{dg}{d\theta} = \sin(c^2\theta^2 + 3) + \theta(2c^2\theta)\cos(c^2\theta^2 + 3)$

(b) $\dfrac{df}{dt} = \dfrac{(t^{2/3} + 5)(2t - 3) - (t^2 - 3t)(2/3)t^{-1/3}}{(t^{2/3} + 5)^2}$

(c) Use implicit differentiation:

$$2y\frac{dy}{dx} + \cos(xy)\left[y + x\frac{dy}{dx}\right] = 0$$

Expand:

$$2y\frac{dy}{dx} + y\cos(xy) + x\cos(xy)\frac{dy}{dx} = 0$$

Solve for $\dfrac{dy}{dx}$ to obtain:

$$\frac{dy}{dx} = -\frac{y\cos(xy)}{2y + x\cos(xy)}$$

Evaluate the derivative at the point $(0, 2)$ to obtain $\dfrac{dy}{dx}\Big|_{(x,y)=(0,2)} = -\dfrac{2\cos(0)}{4 + 0} = -\dfrac{1}{2}$ Thus, we have the point $(0, 2)$ and the slope $-1/2$ to obtain the equation of the tangent line $\boxed{y = -(1/2)x + 2}$

(d) Recall from the Chain Rule that $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$.

i. At $x = 0$ we notice that f' does not exist, since the graph of f has a corner, so $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$ $\boxed{\text{does not exist}}$.

ii. At $x = 1$ we see $f(1) = 0$, $g'(0) = 1$, and $f'(1) = 3$ (we calculate this slope by looking at the rise over the run of the straight line in the graph). Thus, $\frac{d}{dx}g(f(1)) = g'(f(1))f'(1) = \boxed{3}$

iii. At $x = 3$ we have $f(3) = 3$, $g'(3) = 3$ and $f'(3) = 0$, so $\frac{d}{dx}g(f(3)) = g'(f(3))f'(3) = \boxed{0}$

2. (15 points) A conical tank (i.e. an inverted cone) of height $h = 10$ meters and base radius $r = 4$ meters is full of water. The water drains from the bottom of the tank at the rate of 5 m^3/min. How fast is the water level, h, dropping when $h = 6$ meters? (Note: The volume of a cone is given by $V = (1/3)\pi r^2 h$ where r is the radius of the base of the cone and h is the height.)

Solution: Let $h(t)$ be the height, $r(t)$ be the radius, and $V(t)$ be the volume of the water in the tank at time t. We are given that $dV/dt = -5$ meters^3/min. We want to find dh/dt at the instant when $h = 6$ meters.

First, notice that we can use similar triangles to obtain $\frac{10}{4} = \frac{h}{r}$. Thus, $r = \frac{2h}{5}$. Then,

$$\begin{aligned} V &= \frac{\pi}{3}r^2 h = \frac{\pi}{3}\left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3 \\ \frac{dV}{dt} &= \frac{4\pi}{25}h^2\frac{dh}{dt} \\ -5 &= \frac{4\pi}{25}6^2\frac{dh}{dt}|_{h=6} \end{aligned}$$

We then obtain $\frac{dh}{dt}$ evaluated when $h = 6$ is $\boxed{\frac{-125}{144\pi}}$ meters per minute. (Note: The negative indicates that the water level is dropping.)

3. (20 points) Let $g(x) = 6\sqrt{x} - 2x,\ x \geq 0$.

(a) Use the definition of the derivative to show that $g'(x) = \frac{3}{\sqrt{x}} - 2$. (You must use the limit definition of the derivative to earn any credit on this problem.)

(b) Verify that the hypotheses of the Mean Value Theorem are satisfied for g(x) over [0,16]. Then, determine the value(s) c that satisfy the conclusion of the Mean Value Theorem.

Solution:

(a) For $x > 0$ we have:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{6\sqrt{x+h}-2(x+h)-(6\sqrt{x}-2x)}{h} \\
&= \lim_{h\to 0} \left[\frac{6\sqrt{x+h}-6\sqrt{x}}{h} - \frac{2(x+h)-2x}{h}\right] \text{(separate into two fractions)} \\
&= \lim_{h\to 0} \left[6\frac{\sqrt{x+h}-\sqrt{x}}{h} - \frac{2h}{h}\right] \text{(simplify second fraction)} \\
&= \lim_{h\to 0} \left[6\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) - 2\right] \text{(conjugate on first fraction)} \\
&= \lim_{h\to 0} \left[6\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} - 2\right] \\
&= \lim_{h\to 0} \left[6\frac{h}{h(\sqrt{x+h}+\sqrt{x})} - 2\right] \\
&= \lim_{h\to 0} \left[6\frac{1}{\sqrt{x+h}+\sqrt{x}} - 2\right] \\
&= \frac{6}{2\sqrt{x}} - 2 \\
&= \frac{3}{\sqrt{x}} - 2
\end{aligned}$$

Observe that the limit does not exist if $x = 0$ so g is differentiable for $x > 0$.

(b) The two hypotheses for the Mean Value Theorem are (1) g is continuous on $[0, 16]$ since it is the sum of two continuous functions, the square root and a linear function and (2) g is diffferentiable on $(0, 16)$ since we computed the derivative in part (a). Now, we need to find the c value such that

$$g'(c) = \frac{g(16)-g(0)}{16-0} = \frac{-8}{16} = \frac{-1}{2}$$

We have:

$$\frac{3}{\sqrt{c}} - 2 = \frac{-1}{2}$$

Solving, we obtain $\boxed{c=4}$.

4. (27 points) Consider the function $f(x) = x^{\frac{1}{3}}(x+4)$. It has first derivative $f'(x) = \dfrac{4(x+1)}{3x^{\frac{2}{3}}}$ and second derivative $f''(x) = \dfrac{4(x-2)}{9x^{\frac{5}{3}}}$. The following questions will take you step by step through the information you need to sketch a graph of the function.

(a) On what intervals is f increasing? decreasing?

(b) Find the x and y coordinates of the local maximum and minimum values of f, if any exist. If none exist, state this.

(c) On what intervals is f concave up? concave down?

(d) Find the x and y coordinates of the inflection points of f, if any exist. If none exist, state this.

(e) Sketch a graph of f. Carefully label all important points (intercepts, max/min, inflection point(s), etc.)

Solution:

(a) The first derivative has two critical points: $f'(x) = 0$ when $x = -1$ and $f'(x)$ does not exist when $x = 0$.
To find intervals of increasing/decreasing, we plug in values between these critical points.
   i. $f'(-2) < 0$ which indicates that f is decreasing on the interval $(-\infty, -1)$
   ii. $f'(-.5) > 0$ which indicates that f is increasing on $(-1, 0)$
   iii. $f'(1) > 0$ so f is increasing on $(0, \infty)$

(b) From the above and the first derivative test, we know that $f(x)$ has a local minimum at $(-1, -3)$ and no local maxima.

(c) We find that $f''(x) = 0$ at $x = 2$ and $f''(x)$ DNE at $x = 0$. Checking values in between these points, we have
   i. $f''(-1) > 0$ and $f''(3) > 0$ so f is concave up on $(-\infty, 0)$ and $(2, \infty)$
   ii. $f''(1) < 0$ so f is concave down on $(0, 2)$.

(d) By the above, inflection points occur at $(0, 0)$ and $(2, 6\sqrt[3]{2})$.

(e) There are no asymptotes, since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$ and there is no denominator in $f(x)$. The only intercepts are $(0, 0)$ and $(-4, 0)$. Here is the graph (Note that $7.56 \approx 6\sqrt[3]{2}$):