This exam is worth 100 points and has 4 questions.

Start each problem on a new page and put your name at the top of each page.

Show all work and simplify your answers. Name any theorem that you use and explain how it is used. Answers with no justification will receive no points unless the problem explicitly states otherwise

Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to operate your camera for proctoring, view the exam, contact your proctor, or to upload your work.

When you have completed the exam, send a message through chat to your proctor. Your proctor will then give you the ok to scan your exam and upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the zoom proctoring room!

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1. (26 points) The following three problems are not related.

   (a) Suppose $\tan\theta = \frac{5}{2}$ and $\pi < \theta < \frac{3\pi}{2}$. What is the value of $\sec\theta$?

   (b) Find the solutions of $\sin^2 t + 2\sin t = 3$ in $[0, 2\pi)$.

   (c) You are standing some distance away from a tree. The angle of elevation (from the ground where your feet are, to the top of the tree) is $\pi/3$ radians.

   i. How far away are you from the tree if the tree's height is $3\sqrt{3}$ meters?

   ii. What is the distance from your feet to the top of the tree?

Solution:

(a) Use a reference triangle and recall that $\tan\theta = \text{opp}/\text{adj}$. If $\tan\theta = 5/2$, then the opposite side has length 5 and the adjacent side has length 2, then the hypotenuse has length $\sqrt{29}$. In quadrant 3 $\sec\theta = \boxed{-\sqrt{29}/2}$.

(b)

$$\sin^2 t + 2\sin t = 3$$
$$\sin^2 t + 2\sin t - 3 = 0$$
$$(\sin t + 3)(\sin t - 1) = 0$$

Either $\sin t + 3 = 0$ (this has no solution) or $\sin t - 1 = 0$, which occurs when $t = \pi/2$. Thus, the only solution of $\sin^2 t + 2\sin t = 3$ in $[0, 2\pi)$ is $\boxed{t = \pi/2}$.

(c) Let a be the distance from your feet to the tree. We're given that the angle θ of elevation is $\pi/3$. Since

$$\sqrt{3} = \tan\frac{\pi}{3} = \frac{\text{opp}}{\text{adj}} = \frac{\text{height of the tree}}{\text{distance from your feet to the tree}} = \frac{3\sqrt{3}}{a}$$

we have $\sqrt{3} = 3\sqrt{3}/a$. Therefore, $\boxed{a = 3 \text{ meters}}$.

The distance from your feet to the top of the tree is the hypotenuse of the triange. Let s be the length of the hypotenuse. Then $a^2 + (3\sqrt{3})^2 = s^2$. Substitute $a = 3$ and solve for s to find $\boxed{s = 6 \text{ meters}}$.

2. (24 points) Evaluate the limits if they exist. If the limit does not exist, write DNE. Be sure to explain all work. If you use a theorem, please state this and explain how you used it.

   (a) $\displaystyle\lim_{x\to 2} \frac{|x-2|}{x^4 - 4x^2}$

   (b) $\displaystyle\lim_{x\to\infty} (\sqrt{x^2 - x - 1} - x)$

   (c) $\displaystyle\lim_{x\to 0} (x^2 \cos(1/x) + 5)$

   Solution:

   (a) Examine the right-hand and left-hand limits. Note that $|x-2| = \begin{cases} x-2, & x \geq 2 \\ 2-x, & x < 2 \end{cases}$

$$\lim_{x\to 2^+} \frac{|x-2|}{x^4-4x^2} = \lim_{x\to 2^+} \frac{x-2}{x^2(x-2)(x+2)} = 1/16$$
$$\lim_{x\to 2^-} \frac{|x-2|}{x^4-4x^2} = \lim_{x\to 2^-} \frac{2-x}{x^2(x-2)(x+2)} = -1/16$$

   It follows that $\displaystyle\lim_{x\to 2} \frac{|x-2|}{x^4-4x^2}$ $\boxed{\text{does not exist}}$ because the right-hand and left-hand limits are not equal.

   (b) For this problem, we multiply by the conjugate.

$$\begin{aligned}
\lim_{x\to\infty} (\sqrt{x^2-x-1} - x) &= \lim_{x\to\infty} (\sqrt{x^2-x-1} - x)\frac{(\sqrt{x^2-x-1}+x)}{(\sqrt{x^2-x-1}+x)} \\
&= \lim_{x\to\infty} \frac{-x-1}{(\sqrt{x^2-x-1}+x)} \\
&= \lim_{x\to\infty} \frac{-x-1}{(\sqrt{x^2-x-1}+x)} \frac{(1/x)}{(1/x)} \\
&= \lim_{x\to\infty} \frac{-1-1/x}{\sqrt{1-(1/x)-(1/x^2)}+1} = \boxed{\frac{-1}{2}}
\end{aligned}$$

   (c) For this problem, we need to use the Squeeze Theorem. We have

$$\begin{aligned}
-1 \leq &\quad \cos(1/x) &\leq 1 \\
-x^2 \leq &\quad x^2\cos(1/x) &\leq x^2 \\
-x^2 + 5 \leq &\quad x^2\cos(1/x) + 5 &\leq x^2 + 5
\end{aligned}$$

   Since $\displaystyle\lim_{x\to 0}(-x^2+5) = \lim_{x\to 0}(x^2+5) = 5$ we have $\displaystyle\lim_{x\to 0}(x^2\cos(1/x)+5) = 5$ by the Squeeze Theorem.

3. (20 points) Suppose $y = f(x)$ is a function with domain all real numbers.

   (a) Give a short explanation (one or two sentences) of what each of the following conditions tell you about a graph of $y = f(x)$.

i. $\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to-\infty} f(x) = -3$

ii. $\lim_{x\to 2^+} f(x) = -\infty$ and $\lim_{x\to 2^-} f(x) = \infty$

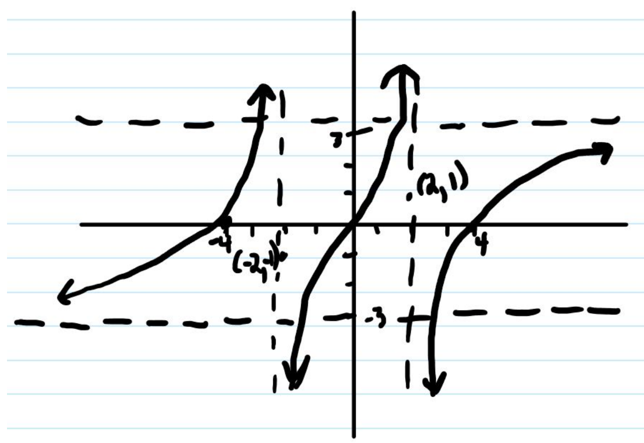
(b) If you also know that $y = f(x)$ is **odd** and if you know that $f(2) = 1$ and $f(4) = 0$, sketch a graph of $y = f(x)$ that satisfies this information and the information given in part (a). Label the axes, asymptotes, and any x or y intercepts.

Solution:

(a) $\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to-\infty} f(x) = -3$ tell us that $y = -3$ and $y = 3$ are horizontal asymptotes.

(b) $\lim_{x\to 2^+} f(x) = -\infty$ and $\lim_{x\to 2^-} f(x) = \infty$ tells us that $x = 2$ is a vertical asymptote.

(c) Since the function is odd with domain all real numbers, we know that (i) $x = -2$ is a vertical asymptote, (ii) $f(-2) = -1$, and (iii) $f(0) = 0$. Putting this together, we get the following:



4. (30 points) The following three questions are unrelated:

(a) Let $f(x) = x + \dfrac{1}{x}$ and $g(x) = \dfrac{3x+5}{x+2}$. Find $(g \circ f)(x)$ and the domain of $(g \circ f)(x)$. Give the domain in interval notation.

(b) Use the definition of continuity (i.e. use the appropriate limits) to find all values of a which make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} x^2 + 4 & x \geq a \\ 4x & x < a \end{cases}$$

(c) Show that $t^3 + \sin t = \frac{1}{t^2+1}$ has at least one solution. If you use a theorem, state this and explain how you used it.

Solution:

(a)

$$\begin{aligned}(g \circ f)(x) &= \frac{3\left(x+\frac{1}{x}\right)+5}{\left(x+\frac{1}{x}\right)+2} \\ &= \frac{3x+\frac{3}{x}+5}{x+\frac{1}{x}+2} \\ &= \frac{3x^2+3+5x}{x^2+1+2x} = \frac{3x^2+5x+3}{(x+1)^2}\end{aligned}$$

and the domain of $(g \circ f)(x)$ therefore includes all real numbers except $x = 0$ (since $f(x)$ isn't defined at 0 and except at $x = -1$ because $(g \circ f)$ is not defined at -1. $D = (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$.

(b) Since both parts of the piecewise functions are polynomials, they are continuous on their domains. The only point of discontinuity could be at a. A function is continuous at a if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Evaluate

$$\lim_{x \to a^-} f(x) = 4a$$

and

$$\lim_{x \to a^+} f(x) = f(a) = a^2 + 4$$

Then, continuity at $x = a$ requires $4a = a^2 + 4$ which yields $\boxed{a = 2}$

(c) This is an Intermediate Value Theorem problem. Define $f(x) = t^3 + \sin t - \frac{1}{t^2+1}$. This function is continuous everywhere, since it is a sum of trigonometric, polynomial, and rational functions and the denominator of the rational function is never 0. Note that $f(0) = -1 < 0$ and $f(3\pi) = 9\pi^2 - \frac{1}{9\pi^2+1} > 0$. By the IVT, there must exist a c such that $-1 < c < 0$ and $f(c) = 0$.

5. At the bottom of your work for problem 4: please write a statement that says your work is your own and you did not receive any help on the exam. Sign this statement.

Trigonometric identities

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$