1. (32 pts) Evaluate the following. Simplify your answers.

   (a) $\lim_{x\to 0} \dfrac{5x^{40} - 8x^{20}}{10x^{40} + 7x^{20}}$

   (b) $\lim_{x\to 0^+} (\sin x)^{\pi/(\ln x)}$

   (c) Let $f(t) = \arctan\left(3^t\right)$. Find $f'(1)$.

   (d) Let $g(t) = (1+2t)^{\cos t}$. Find $g'(\pi)$.

Solution:

(a) $\displaystyle \lim_{x\to 0} \frac{5x^{40} - 8x^{20}}{10x^{40} + 7x^{20}} = \lim_{x\to 0} \frac{\cancel{x^{20}}\left(5x^{20} - 8\right)}{\cancel{x^{20}}\left(10x^{20} + 7\right)} = \frac{0-8}{0+7} = \boxed{-\frac{8}{7}}.$

(b)

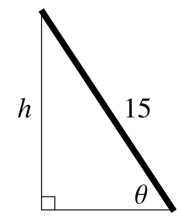
$$\begin{aligned} y &= \lim_{x\to 0^+} (\sin x)^{\pi/(\ln x)} \\ \ln y &= \lim_{x\to 0^+} \ln\left((\sin x)^{\pi/(\ln x)}\right) \\ &= \lim_{x\to 0^+} \frac{\pi \ln(\sin x)}{\ln x} \\ &\overset{LH}{=} \lim_{x\to 0^+} \frac{\pi \cdot \frac{\cos x}{\sin x}}{\frac{1}{x}} \\ &= \lim_{x\to 0^+} \pi \cdot \frac{x}{\sin x} \cdot \cos x = \pi \cdot 1 \cdot 1 = \pi \\ y &= \boxed{e^{\pi}} \end{aligned}$$

(c) $\displaystyle f(t) = \arctan\left(3^t\right) \;\Rightarrow\; f'(t) = \frac{3^t(\ln 3)}{1+3^{2t}} \;\Rightarrow\; f'(1) = \boxed{\frac{3\ln 3}{10}}.$

(d) Use logarithmic differentiation.

$$\begin{aligned} y &= (1+2t)^{\cos t} \\ \ln y &= \ln\left((1+2t)^{\cos t}\right) \\ &= (\cos t)\ln(1+2t) \\ \frac{1}{y}\frac{dy}{dt} &= (\cos t)\frac{2}{1+2t} - (\sin t)\ln(1+2t) \\ \frac{dy}{dt} &= (1+2t)^{\cos t}\left(\frac{2\cos t}{1+2t} - (\sin t)\ln(1+2t)\right) \\ g'(\pi) = \left.\frac{dy}{dt}\right|_{t=\pi} &= (1+2\pi)^{\cos\pi}\left(\frac{2\cos\pi}{1+2\pi} - \cancel{(\sin\pi)\ln(1+2\pi)}\right) \\ &= (1+2\pi)^{-1}\left(\frac{-2}{1+2\pi}\right) = \boxed{\frac{-2}{(1+2\pi)^2}} \end{aligned}$$

2. (12 pts) A ladder 15 ft long propped against a wall forms an angle of θ radians with the ground. The top of the ladder begins to slide down the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is θ changing when the top of the ladder is $h = 12$ ft above the ground?



Solution: Given: $dh/dt = -1/4$ ft/sec. Find $d\theta/dt$ when $h = 12$ ft.

At this moment the base of the ladder is $\sqrt{15^2 - 12^2} = 9$ ft from the wall.

$$\begin{aligned}
\sin\theta &= \frac{h}{15} \\
\cos\theta \, \frac{d\theta}{dt} &= \frac{1}{15} \cdot \frac{dh}{dt} \\
\frac{9}{15} \cdot \frac{d\theta}{dt} &= \frac{1}{15}\left(-\frac{1}{4}\right) \\
\frac{d\theta}{dt} &= \frac{\cancel{15}}{9} \cdot \frac{1}{\cancel{15}}\left(-\frac{1}{4}\right) = \boxed{-\frac{1}{36} \text{ rad/sec}}.
\end{aligned}$$

Alternate Solution:

$$\begin{aligned}
\theta &= \arcsin\left(\frac{h}{15}\right) \\
\frac{d\theta}{dt} &= \frac{1}{\sqrt{1 - \frac{h^2}{15^2}}} \cdot \frac{1}{15} \cdot \frac{dh}{dt} \\
&= \frac{1}{\sqrt{1 - \frac{12^2}{15^2}}} \cdot \frac{1}{15}\left(-\frac{1}{4}\right) = \boxed{-\frac{1}{36} \text{ rad/sec}}.
\end{aligned}$$

3. (24 pts) Evaluate the following. Simplify your answers.

(a) $\displaystyle\int_{1/2}^{1} 8x^{-2}\left(1 + \frac{1}{x}\right)^{-3} dx$

(b) $\displaystyle\int \frac{s^2}{\sqrt{1 - s^6}}\, ds$

(c) Let $f(x) = 10 + \displaystyle\int_9^{x^2} \sin\left(\frac{\pi\sqrt{t}}{2}\right) dt$. Find $f'(3)$.

Solution:

(a) Let $u = 1 + 1/x$, $du = -x^{-2}$. Then the u-limits are 3 to 2.

$$\int_{1/2}^{1} 8x^{-2}\left(1+\frac{1}{x}\right)^{-3} dx = \int_3^2 -8u^{-3}\,du = -8\cdot\left.\frac{u^{-2}}{-2}\right|_3^2 = 4u^{-2}\Big|_3^2$$

$$= 4\left(\frac{1}{4}-\frac{1}{9}\right) = 4\cdot\frac{5}{36} = \boxed{\frac{5}{9}}.$$

(b) Let $u = s^3$, $du = 3s^2\,ds$.

$$\int \frac{s^2}{\sqrt{1-s^6}}\,ds = \int \frac{1}{3}\frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}u + C = \boxed{\frac{1}{3}\sin^{-1}\left(s^3\right)+C} = \boxed{-\frac{1}{3}\cos^{-1}\left(s^3\right)+C}.$$

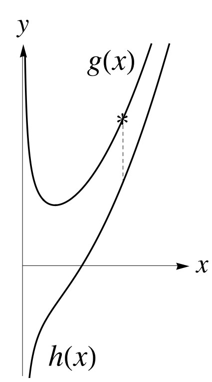
(c) Use the FTC-1 and the chain rule.

$$f(x) = 10 + \int_9^{x^2} \sin\left(\frac{\pi\sqrt{t}}{2}\right) dt.$$

$$f'(x) = \sin\left(\frac{\pi\sqrt{x^2}}{2}\right)(2x).$$

$$f'(3) = \sin\left(\frac{3\pi}{2}\right)(2\cdot 3) = \boxed{-6}.$$

4. (12 pts) Buff Bug is crawling along the curve defined by $g(x) = x^2 + \frac{1}{\sqrt{x}}$, $x > 0$, and wishes to move down to a lower curve $h(x) = x^2 - \frac{x}{8} - \frac{1}{\sqrt{x}}$ using the shortest vertical route possible. At what x-coordinate should Buff Bug descend?



Solution: We wish to minimize the distance $d(x) = g(x) - h(x)$.

$$d(x) = x^2 + \frac{1}{\sqrt{x}} - \left(x^2 - \frac{x}{8} - \frac{1}{\sqrt{x}}\right)$$

$$= \frac{x}{8} + \frac{2}{\sqrt{x}}$$

$$d'(x) = \frac{1}{8} - \frac{1}{x^{3/2}}$$

Find the critical numbers.

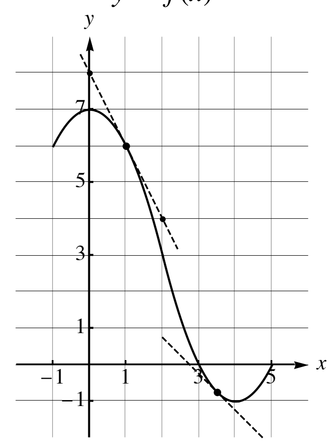
$$d'(x) = 0 \Rightarrow \frac{1}{8} = \frac{1}{x^{3/2}} \Rightarrow x^{3/2} = 8 \Rightarrow x = 8^{2/3} = 4.$$

Because $d''(x) = \frac{3}{2}x^{-5/2} > 0$ for $x > 0$, the function $d(x)$ is concave up and has a minimum value at $\boxed{x = 4}$.

5. (36 pts) The continuous and differentiable function $f(x)$, shown at right, is defined for $-1 \le x \le 5$ and has the following properties.

   - The tangent line at $x = \frac{7}{2}$ is $y = -\frac{3}{4} - \left(x - \frac{7}{2}\right)$.
   - The tangent line at $(1, 6)$ is shown.
   - $\displaystyle\int_{-1}^{3} f(x)\,dx = \frac{58}{3}$.
   - $\displaystyle\int_{-1}^{5} f(x)\,dx = \frac{54}{3} = 18$.

$y = f(x)$



Answer the following questions about $f(x)$. Justify your work.

(a) Approximate $\displaystyle\int_{-1}^{5} |f(x)|\,dx$ using an upper sum with three equal subintervals.

(b) Find the exact value of $\displaystyle\int_{-1}^{5} |f(x)|\,dx$.

(c) Use the linearization of f at $x = 1$ to approximate $f(0.99)$.

(d) If Newton's Method is applied to $f(x)$ with an initial approximation of $x_1 = \frac{7}{2}$, what is the value of x_2?

(e) Find one of the c values that satisfies the conclusion of the Mean Value Theorem for Derivatives on the interval $[-1, 5]$.

(f) Find the c value that satisfies the conclusion of the Mean Value Theorem for Integrals on the interval $[-1, 5]$.

Solution:

(a) Let $\Delta x = \frac{6}{3} = 2$. Then $\displaystyle\int_{-1}^{5} |f(x)|\,dx \approx \Delta x\,(f(0) + f(1) + |f(4)|) = 2(7 + 6 + 1) = \boxed{28}$.

(b) $\displaystyle\int_{3}^{5} f(x)\,dx = \int_{-1}^{5} f(x)\,dx - \int_{-1}^{3} f(x)\,dx = \frac{54}{3} - \frac{58}{3} = -\frac{4}{3}$.

$\displaystyle\int_{-1}^{5} |f(x)|\,dx = \int_{-1}^{3} f(x)\,dx - \int_{3}^{5} f(x)\,dx = \frac{58}{3} - \left(-\frac{4}{3}\right) = \boxed{\frac{62}{3}}$.

(c) From the graph, $f(1) = 6$ and $f'(1) = -2$.
Then the linearization at $x = 1$ is $L(x) = f(1) + f'(1)(x - 1) = 6 - 2(x - 1)$.
An approximation for $f(0.99)$ is $L(0.99) = 6 - 2(0.99 - 1) = \boxed{6.02}$.

(d) The tangent line at $x_1 = \frac{7}{2}$ is $y = -\frac{3}{4} - \left(x - \frac{7}{2}\right)$. The next approximation x_2 equals the x-intercept of this line.

$$-\tfrac{3}{4} - \left(x - \tfrac{7}{2}\right) = 0 \Rightarrow x - \tfrac{7}{2} = -\tfrac{3}{4} \Rightarrow x = \tfrac{7}{2} - \tfrac{3}{4} = \boxed{\tfrac{11}{4}}.$$

Alternate Solution:

The next approximation is $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = \frac{7}{2} - \dfrac{-\frac{3}{4}}{-1} = \frac{7}{2} - \frac{3}{4} = \boxed{\frac{11}{4}}$.

(e) The Mean Value Theorem for Derivatives states that there is at least one value of c in $(-1, 5)$ such that

$$f'(c) = \frac{f(5) - f(-1)}{5 - (-1)} = \frac{0 - 6}{6} = -1.$$

The tangent line at $x = \frac{7}{2}$ has a slope of -1 so $c = \boxed{\frac{7}{2}}$ is one such value.

(f) The Mean Value Theorem for Integrals states that there is at least one value of c in $[-1, 5]$ such that $f(c) = f_{ave} = \frac{1}{6}\int_{-1}^{5} f(x)\,dx = \frac{1}{6}(18) = 3$. Based on the graph, $f(2) = 3$ so $c = \boxed{2}$.

6. (20 pts) The following two problems are not related.

(a) Find the inverse of the function $y = \dfrac{\cosh x}{e^x}$. (You may assume that y is one-to-one.)

(b) A scientist is working with two radioactive isotopes, A and B. Her two samples of A and B had the same mass m_0 initially but their relative decay rates (constants k_A and k_B, respectively) differ.

i. Find expressions for $m_A(t)$ and $m_B(t)$, the mass of each sample at time t.

ii. The ratio of the two masses, $r(t) = m_A(t)/m_B(t)$, satisfies the differential equation $dr/dt = \alpha r$ for some constant value of α. Find α in terms of k_A and k_B.

Solution:

(a) Apply the definition of $\cosh x$, solve for x, then swap x and y.

$$\begin{aligned} y &= \frac{\cosh x}{e^x} = \frac{e^x + e^{-x}}{2} \cdot \frac{1}{e^x} \\ y &= \frac{1 + e^{-2x}}{2} \\ 1 + e^{-2x} &= 2y \\ e^{-2x} &= 2y - 1 \\ -2x &= \ln(2y - 1) \\ x &= -\frac{1}{2}\ln(2y - 1) \\ y^{-1} &= \boxed{-\frac{1}{2}\ln(2x - 1)} \end{aligned}$$

(b) i. $\boxed{m_A(t) = m_0 e^{k_A t}}$ and $\boxed{m_B(t) = m_0 e^{k_B t}}$.

ii.

$$\begin{aligned} r(t) &= \frac{m_A(t)}{m_B(t)} = \frac{\cancel{m_0} e^{k_A t}}{\cancel{m_0} e^{k_B t}} = \frac{e^{k_A t}}{e^{k_B t}} = e^{(k_A - k_B)t} \\ \frac{dr}{dt} &= (k_A - k_B)\, e^{(k_A - k_B)t} \\ &= (k_A - k_B)\, r(t). \end{aligned}$$

The constant α equals $\boxed{k_A - k_B}$.

7. (14 pts) Sketch a graph of a single function $y = g(x)$ with all of the following properties. No justification is necessary.

- $\lim_{x \to a} g(x) = g(a)$ for all $a \neq -2$
- $\lim_{x \to -2} g(x) = \infty$
- $\lim_{x \to -\infty} g(x) = 2$
- $g(0) = -2$
- $\int_0^2 g(x)\, dx = 0$
- $\lim_{h \to 0} \frac{g(h) - g(0)}{h} = 2$
- $\lim_{x \to a} \frac{g(x) - g(a)}{x - a} < 0$ for $a > 2$

Solution:

- g is continuous except at $x = -2$.
- g approaches ∞ on both sides of $x = -2$.
- there is a horizontal asymptote at $y = 2$ as g approaches $-\infty$.
- $g(0) = -2$
- $\int_0^2 g(x)\, dx = 0$
- $f'(0) = 2$, there should not be a corner at $x = 0$.
- g is decreasing for $x > 2$.

Here is one possible solution.