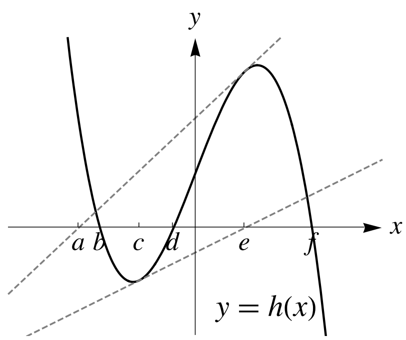
1. (26 pts) The following problems are not related.

   (a) Let $f(x) = \dfrac{2x^3 - 3x^2 + 2x}{x^2+1}$. Find the slant asymptote of $y = f(x)$. Justify your answer using limits. (*Similar to HW 8 #1*)

   (b) Suppose Newton's Method is used to approximate a root of $h(x)$. Tangent lines to the curve $y = h(x)$ corresponding to the first two approximations are shown. Match the first three approximations x_1, x_2, and x_3 to the x-coordinates (a, b, c, d, e, or f). No justification is necessary. (*Similar to Fall 2018 Exam 3 #3c*)



   (c) Find the positive integer n that satisfies $\displaystyle\sum_{i=1}^{n}(2i+4) = 4n + 9900$. (*Similar to Fall 2018 Exam 3 #3b*)

   (d) Suppose g is an odd function continuous on $[-7, 7]$. Given

$$\int_{-3}^{7} g(x)\,dx = 10 \quad \text{and} \quad \int_{-7}^{0} g(x)\,dx = 9,$$

   find $\displaystyle\int_0^3 g(x)\,dx$.

Solution:

   (a) Using polynomial long division we find that

$$\begin{array}{r|l} & 2x - 3 \\ \hline x^2+1 & 2x^3 - 3x^2 + 2x \\ & -2x^3 \qquad\quad - 2x \\ \hline & -3x^2 \\ & 3x^2 \qquad + 3 \\ \hline & 3 \end{array}$$

   so $f(x) = 2x - 3 + \dfrac{3}{x^2+1}$. We can verify that there is a slant asymptote at $\boxed{y = 2x-3}$.

$$\lim_{x\to\infty}[f(x) - (mx+b)] = \lim_{x\to\infty}\left[2x - 3 + \frac{3}{x^2+1} - (2x-3)\right] = \lim_{x\to\infty}\frac{3}{x^2+1} = 0.$$

(b) $\boxed{c, e, a}$

(c)

$$\begin{aligned}\sum_{i=1}^{n}(2i+4) &= 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 4\\ &= 2\cdot\frac{n(n+1)}{2} + 4n\\ &= n^2+5n = 4n+9900\\ n^2+n = n(n+1) &= 9900\end{aligned}$$

If two consecutive positive integers have a product of 9900, the smaller number must be 99. We can verify by solving the quadratic equation.

$$\begin{aligned}n^2+n-9900 &= 0\\ (n-99)(n+100) &= 0\\ n &= 99, -100\end{aligned}$$

We are given that $n > 0$ so $n = \boxed{99}$.

(d) If f is an odd function then $\int_{-a}^{a} f(x)\,dx = 0$ and $\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$.
We are given that $\int_{-7}^{0} f(x)\,dx = 9$ so $\int_{0}^{7} f(x)\,dx = -9$.
It follows that $\int_{-3}^{0} f(x)\,dx = \int_{-3}^{7} f(x)\,dx - \int_{0}^{7} f(x)\,dx = 10 - (-9) = 19$.
Therefore $\int_{0}^{3} f(x)\,dx = \boxed{-19}$.

Alternate Solution:
We are given that $\int_{-7}^{0} f(x)\,dx = 9$ so $\int_{0}^{7} f(x)\,dx = -9$.
We are also given that $\int_{-3}^{7} f(x)\,dx = 10$. Since $\int_{-3}^{3} f(x)\,dx = 0$, then $\int_{3}^{7} f(x)\,dx = 10$.
Then $\int_{0}^{3} f(x)\,dx = \int_{0}^{7} f(x)\ - \int_{3}^{7} f(x)\ = -9 - 10 = \boxed{-19}$.

2. (14 pts) Suppose you want to make an open-top box out of a square sheet of cardboard by cutting out a small square from each corner and bending up the sides. If the cardboard sheet measures 1 meter by 1 meter, what is the largest volume that such a box can have? (*Similar to WA 3.4-3.5 #4*)

Solution: Let the side length of each small square be x meters. Then we can represent the side length of the base of the completed box as $1-2x$ and the height of the box as x. Since we must cut out one square from each corner, we also know that $0 \le x \le 1/2$.

We can represent the volume of the box as $V(x) = x(1-2x)^2$. We can then find the derivative of the volume in terms of the size of the small squares by first rewriting the volume equation and then taking the derivative with respect to x.

$$\begin{aligned}V(x) &= x(1-2x)^2 = x(1-4x+4x^2) = 4x^3-4x^2+x\\ \frac{dV}{dx} &= 12x^2-8x+1 \quad \big(\text{or by the product rule: } -4x(1-2x)+(1-2x)^2\big)\end{aligned}$$

Since we want to maximize this volume, we find the points where $dV/dx = 0$. We can rewrite

$$\frac{dV}{dx} = 12x^2-8x+1 = (2x-1)(6x-1)$$

which equals zero when $x = \frac{1}{2}$ or $\frac{1}{6}$. Furthermore, based on the domain of x, we note that $x = \frac{1}{2}$ is an endpoint and $V\left(\frac{1}{2}\right) = 0$. Next we apply the second derivative test to determine whether there is a maximum or minimum value at $x = \frac{1}{6}$.

$$\frac{d^2V}{dx^2} = 24x - 8$$
$$\left.\frac{d^2V}{dx^2}\right|_{x=1/6} = 24\left(\tfrac{1}{6}\right) - 8 = -4.$$

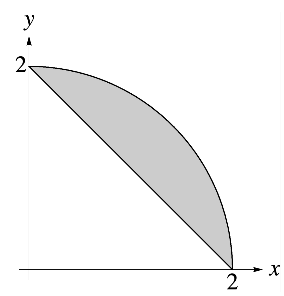
Since $\frac{d^2V}{dx^2} < 0$ when $x = \frac{1}{6}$, we know that $x = \frac{1}{6}$ is a maximum. Since $V(0) = V(\frac{1}{2}) = 0$, we also know that it is the absolute maximum. Thus, the max volume occurs when the side length of the small squares is $x = \frac{1}{6}$. The maximal volume of the box is therefore $V\left(\frac{1}{6}\right) = \frac{1}{6}\left(1 - 2\cdot\frac{1}{6}\right)^2 = \boxed{\frac{2}{27}\text{ m}^3}$.

3. (24 pts) The following problems are not related.

   (a) Find $g(\theta)$ given $g''(\theta) = 2\sin\theta + \cos\theta,\ g(0) = \pi$, and $g\left(\frac{\pi}{2}\right) = -1$. (*Similar to WA 3.7 #1*)

   (b) Evaluate $\int x^{-1/2}\sec^5(\sqrt{x})\tan(\sqrt{x})\,dx$. (*Similar to WA 4.5 #3*)

   (c) Given a circle of radius 2 in the first quadrant, write down an integral (or integrals) to represent the area of the shaded region shown below, then evaluate your integral(s) using geometry. (*Similar to HW 10 #2*)



Solution:

(a) First find $g(\theta)$ in terms of arbitrary constants C and D.

$$g''(\theta) = 2\sin\theta + \cos\theta$$
$$g'(\theta) = -2\cos\theta + \sin\theta + C$$
$$g(\theta) = -2\sin\theta - \cos\theta + C\theta + D$$

Next use the initial values $g(0) = \pi$ and $g\left(\frac{\pi}{2}\right) = -1$ to solve for C and D.

$$g(0) = -2\sin 0 - \cos 0 + C(0) + D = -1 + D = \pi \ \Rightarrow\ D = \pi + 1$$
$$g\left(\tfrac{\pi}{2}\right) = -2\sin\tfrac{\pi}{2} - \cos\tfrac{\pi}{2} + \tfrac{\pi}{2}C + D = -2 + \tfrac{\pi}{2}C + \pi + 1 = -1 \ \Rightarrow\ C = -2$$

The solution is $g(\theta) = \boxed{-2\sin\theta - \cos\theta - 2\theta + \pi + 1}$.

(b) Let $u = \sec(\sqrt{x}),\ \ du = \frac{1}{2}x^{-1/2}\sec(\sqrt{x})\tan(\sqrt{x})\,dx$.

$$\int x^{-1/2}\sec^5(\sqrt{x})\tan(\sqrt{x})\,dx = \int 2u^4\,du = \tfrac{2}{5}u^5 + C = \boxed{\tfrac{2}{5}\sec^5(\sqrt{x}) + C}.$$

Alternate Solution:

Let $u = \cos(\sqrt{x})$, $\; du = -\frac{1}{2}x^{-1/2}\sin(\sqrt{x})\,dx$.

$$\int x^{-1/2}\sec^5(\sqrt{x})\tan(\sqrt{x})\,dx = \int x^{-1/2}\frac{\sin(\sqrt{x})}{\cos^6(\sqrt{x})}\,dx$$
$$= \int -2u^{-6}\,du = \int \tfrac{2}{5}u^{-5} + C = \boxed{\tfrac{2}{5}\left(\cos(\sqrt{x})\right)^{-5} + C}$$

(c) The shaded region corresponds to a quarter-circle of radius 2 minus an isosceles right triangle bounded by the line $y = 2 - x$.

$$A = \int_0^2 \left(\sqrt{4 - x^2} - (2 - x)\right)dx$$

We can find the area of the shaded region using the geometry formulas πr^2 for the circle and $\frac{1}{2}bh$ for the triangle.

$$A = \tfrac{1}{4}\left(\pi \cdot 2^2\right) - \tfrac{1}{2}2^2 = \boxed{\pi - 2}$$

4. (14 pts) Ralphie travels at a velocity of $v(t) = t^2 - 3t + 2$ starting at $t = 0$ until $t = 2$. (*Similar to HW 11 #4*)

(a) Find Ralphie's average velocity on $[0, 2]$.

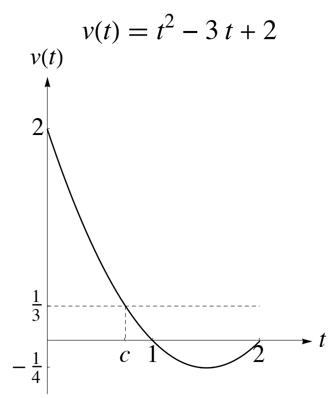
(b) Sketch a graph of $v(t)$ and mark the approximate location of the number c that satisfies the Mean Value Theorem for Integrals. Justify your answer. (It is not necessary to find the exact value of c.)

Solution:

(a) The average velocity is

$$\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{2}\int_0^2 \left(t^2 - 3t + 2\right)dt$$
$$= \frac{1}{2}\left[\frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t\right]_0^2 = \frac{1}{2}\left(\frac{8}{3} - 6 + 4\right)$$
$$= \frac{1}{2}\left(\frac{2}{3}\right) = \boxed{\frac{1}{3}}.$$

(b) By the Mean Value Theorem for Integrals there is a number c in $[0, 2]$ such that $v(c)$ equals the average velocity of $\frac{1}{3}$.



5. (22 pts) The following two problems are not related.

(a) Consider the curve $y = \dfrac{x}{(x^2+3)^2}$, $1 \le x \le 3$. (*Similar to HW 10 #5*)

i. Find an expression for R_n, the Riemann sum using right endpoints and n equal subintervals. Do not evaluate the expression.

ii. Find $\lim_{n\to\infty} R_n$ by evaluating an integral.

(b) Let

$$g(x) = 2 + \int_{x^2}^{4} \cos\left(\frac{\pi\sqrt{t}}{2}\right) dt, \quad 0 \le x \le 3.$$

Find the linearization of g at $x = 2$. (*Similar to HW 11 #2*)

Solution:

(a) i. Let $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $x_i = a + i\Delta x = 1 + \frac{2i}{n}$.

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \boxed{\sum_{i=1}^{n} \frac{1+\frac{2i}{n}}{\left(\left(1+\frac{2i}{n}\right)^2+3\right)^2}\left(\frac{2}{n}\right)}.$$

ii.

$$\lim_{n\to\infty} R_n = \int_1^3 \underbrace{\frac{x}{(x^2+3)^2}\,dx}_{\substack{u=x^2+3\\ du=2x\,dx}} = \int_4^{12} \frac{1}{2}u^{-2}\,du = \frac{1}{2}\left[-\frac{1}{u}\right]_4^{12} = \frac{1}{2}\left(-\frac{1}{12}+\frac{1}{4}\right) = \boxed{\frac{1}{12}}.$$

(b)

$$g(2) = 2 + \int_4^4 \cos\left(\frac{\pi\sqrt{t}}{2}\right) dt = 2 + 0 = 2$$

$$g'(x) = -2x\cos\left(\frac{\pi x}{2}\right)$$

$$g'(2) = -4\cos(\pi) = 4$$

$$L(x) = g(2) + g'(2)(x-2) = \boxed{2 + 4(x-2)} = 4x - 6$$