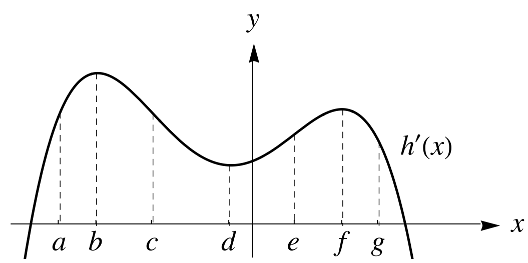
1. (30 pts) The following three problems are not related.

   (a) No justification is necessary for the following questions about the graph of <u>derivative</u> $h'(x)$ shown below. At which of the x-coordinates (a, b, c, d, e, f, or g) is

      i. $h(x)$ greatest?
      ii. $h(x)$ least?
      iii. $h'(x)$ greatest?
      iv. $h'(x)$ least?
      v. $h''(x)$ greatest?
      vi. $h''(x)$ least?



   (b) Let $y = \sqrt{x^3 - \sin(\pi x)}$. Find an equation of the line tangent to y at $x = 1$.

   (c) Find dy/dx if $\cos(3xy) = (y-1)^2$.

Solution:

(a) $\boxed{g, a, b, d, a, g}$

(b) We have

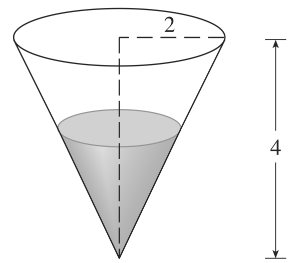
$$y' = \frac{1}{2}\left(x^3 - \sin(\pi x)\right)^{-1/2}\left(3x^2 - \pi\cos(\pi x)\right)$$
$$y'(1) = \frac{1}{2}(1)(3+\pi) = \frac{3+\pi}{2}$$
$$y(1) = 1$$

An equation of the tangent line is therefore $\boxed{y = 1 + \left(\frac{3+\pi}{2}\right)(x-1)}$.

(c)

$$\cos(3xy) = (y-1)^2$$
$$-\sin(3xy)\cdot\left(3y + 3x\frac{dy}{dx}\right) = 2(y-1)\frac{dy}{dx}$$
$$-3y\sin(3xy) - 3x\sin(3xy)\frac{dy}{dx} = 2(y-1)\frac{dy}{dx}$$
$$-2(y-1)\frac{dy}{dx} - 3x\sin(3xy)\frac{dy}{dx} = 3y\sin(3xy)$$
$$\frac{dy}{dx} = \boxed{\frac{3y\sin(3xy)}{-2(y-1) - 3x\sin(3xy)}}$$
$$= \frac{-3y\sin(3xy)}{2(y-1) + 3x\sin(3xy)}$$

2. (15 pts) A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being drained from the tank at a rate of π m^3/min, how deep is the water when the water level is falling at 9 m/min?



Solution: Let h, r, and V represent the height, radius, and volume of the water, respectively. We are given that $dV/dt = -\pi$ m^3/min. The ratio r/h matches the proportions of the cone: $r/h = 2/4 = 1/2$ so $r = h/2$. We wish to find h when $dh/dt = -9$ m/min.

$$\begin{aligned} V &= \frac{\pi}{3}r^2 h = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3 \\ \frac{dV}{dt} &= \frac{\pi}{4}h^2\frac{dh}{dt} \\ -\pi &= \frac{\pi}{4}h^2(-9) \\ \frac{4}{9} &= h^2 \Rightarrow \boxed{h = \frac{2}{3}\text{ m}} \end{aligned}$$

Alternate Solution: Use the fact that $r = h/2 \Rightarrow dr/dt = \frac{1}{2}dh/dt$.

$$\begin{aligned} V &= \frac{\pi}{3}r^2 h \\ \frac{dV}{dt} &= \frac{\pi}{3}r^2\frac{dh}{dt} + \frac{2\pi}{3}rh\frac{dr}{dt} \\ \frac{dV}{dt} &= \frac{\pi}{3}\left(\frac{h}{2}\right)^2\frac{dh}{dt} + \frac{2\pi}{3}\left(\frac{h}{2}\right)h\left(\frac{1}{2}\cdot\frac{dh}{dt}\right) \\ -\pi &= \frac{\pi}{12}h^2(-9) + \frac{\pi}{6}h^2(-9) \\ -1 &= -\frac{3}{4}h^2 - \frac{3}{2}h^2 \\ 1 &= \frac{9}{4}h^2 \Rightarrow h = \frac{2}{3}\text{ m} \end{aligned}$$

3. (20 pts) The following two problems are not related.

   (a) Compute an approximation to $\sqrt{1.1}$ by finding a linearization for $g(x) = \sqrt{1+x}$ at $a = 0$.

   (b) When a guitar string is plucked, it vibrates with a frequency

$$f(T) = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

   where T is the tension, L is the length of the string, and μ is the linear mass density. "Tuning" the guitar means adjusting the tension while L and μ stay constant, so that the desired note is produced when plucked. If a musician wants to make sure that her string vibrates so that the relative error in the frequency f is less than 0.01, then what is the largest acceptable relative error in the tension T?

Solution:

   (a) The linearization function is $L(x) = g(a) + g'(a)(x - a)$ for $a = 0$.

$$\begin{aligned}
&g(x) = \sqrt{1+x} \quad \text{and} \quad g'(x) = \frac{1}{2\sqrt{1+x}} \\
&g(0) = 1 \quad \text{and} \quad g'(0) = \frac{1}{2} \\
&L(x) = g(0) + g'(0)(x) = 1 + \frac{1}{2}x \\
&\sqrt{1.1} = g(0.1) \approx L(0.1) = 1 + \frac{1}{2}(0.1) = \boxed{1.05}
\end{aligned}$$

   (b) We are given that the relative error in frequency df/f is less than 0.01. We wish to find the corresponding relative error in tension dT/T.

$$\begin{aligned}
f(T) &= \frac{1}{2L}\sqrt{\frac{T}{\mu}} \\
df &= \frac{1}{2L\sqrt{\mu}} \cdot \frac{dT}{2\sqrt{T}} = \frac{dT}{4L\sqrt{\mu}\sqrt{T}} \\
\frac{df}{f} &= \frac{\frac{dT}{4L\sqrt{\mu}\sqrt{T}}}{\frac{1}{2L\sqrt{\mu}}\sqrt{T}} = \frac{dT}{2T} \\
\frac{df}{f} &= \frac{dT}{2T} < 0.01 \Rightarrow \frac{dT}{T} < \boxed{0.02}
\end{aligned}$$

4. (15 pts) Let $f(x) = x + 2\cos(x)$ on $[0, 2\pi]$.

(a) On what intervals is f increasing? decreasing?

(b) Find the (x, y) coordinates of the local maximum and minimum values of f, if any.

(c) Find the (x, y) coordinates of the absolute maximum and minimum values of f.

$\left(\textit{Hint:} \sqrt{2} \approx 1.4 \text{ and } \sqrt{3} \approx 1.7\right)$

Solution:

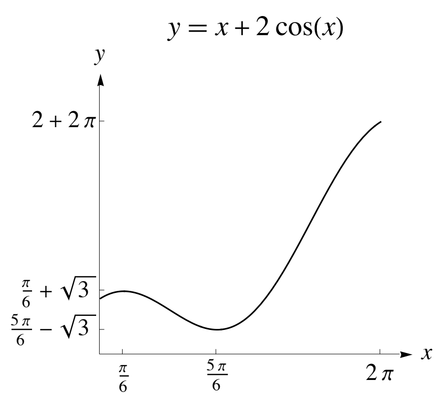
(a) $f'(x) = 1 - 2\sin x$

$f'(x) = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$.

Intervals	y'	y
$0 \le x < \frac{\pi}{6}$	+	increasing
$\frac{\pi}{6} < x < \frac{5\pi}{6}$	−	decreasing
$\frac{5\pi}{6} < x \le 2\pi$	+	increasing

(b) There is a local maximum value at $\boxed{\left(\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3}\right)}$ and a local minimum value at $\boxed{\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)}$.

(c) The endpoints are located at $(0, 2)$ and $(2\pi, 2\pi + 2)$. There is an absolute minimum value at $\boxed{\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)}$ and an absolute maximum value of $\boxed{(2\pi, 2\pi + 2)}$.



5. (20 pts) The following two problems are not related.

(a) Verify that $g(x) = x^3 - 3x^2$ satisfies Rolle's theorem on $[0, 3]$ and find all x-values whose existence is guaranteed by Rolle's theorem on this interval.

(b) Given

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

use the definition of derivative to determine if $f(x)$ is differentiable at $x = 0$. You may assume that f is continuous at $x = 0$.

Solution:

(a) $g(x)$ is a polynomial, hence it continuous on $[0, 3]$ and differentiable on $(0, 3)$.
Moreover, $g(0) = 0$ and $g(3) = 3^3 - 3 \cdot 3^2 = 27 - 27 = 0$.
Thus g satisfies the conditions of Rolle's theorem on $[0, 3]$.
By Rolle's theorem there must be at least one x-value in the interval $(0, 3)$ for which $g'(x) = 0$.
But $g'(x) = 3x^2 - 6x = 3x(x - 2)$ so $g'(x) = 0$ when $x = 0$ or $x = 2$. But 0 is not in the interval $(0, 3)$, so $\boxed{x = 2}$ is the special x-value in $(0, 3)$ whose existence is guaranteed by Rolle's theorem.

(b) We investigate whether

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

exists. But as $h \to 0^+$ (or as $h \to 0^-$) $\sin\left(\frac{1}{h}\right)$ oscillates between -1 and 1 more and more often forever, meaning that both one-sided limits do not exist. Thus the limit in the definition of $f'(0)$ does not exist, meaning that $f(x)$ is $\boxed{\text{not differentiable}}$ at $x = 0$.