1. (15 pts) The following two problems are not related.

   (a) Solve the inequality on \([0, 2\pi]\) and write your answer in interval notation.

   \[
   2 \cos x \sec x \sin x + 1 < 0
   \]

   \[
   2 \sin x + 1 < 0
   \]

   \[
   \sin x < -\frac{1}{2}
   \]

   The inequality is satisfied when \(x\) is in \(\left(\frac{7\pi}{6}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)\). Note that \(\sec x\) is undefined at \(x = \frac{3\pi}{2}\).

   (b) A 20-ft ladder leaning against a wall forms an angle of \(\theta\) with the wall, with a maximum angle of \(\frac{\pi}{3}\) radians. For larger angles the base of the ladder slips and the ladder falls to the ground. What is the slope of the ladder at this maximum angle?

   Solution:

   (a)

   \[
   2 \cos x \sec x \sin x + 1 < 0
   \]

   \[
   2 \sin x + 1 < 0
   \]

   \[
   \sin x < -\frac{1}{2}
   \]

   The inequality is satisfied when \(x\) is in \(\left(\frac{7\pi}{6}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)\). Note that \(\sec x\) is undefined at \(x = \frac{3\pi}{2}\).

   (b) The slope of the ladder is \(y/x < 0\). When \(\theta = \pi/3\) the slope is \(-\cot \frac{\pi}{3} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}\).
2. (24 pts) Evaluate the following limits and simplify your answers.

(Reminder: You may not use L’Hopital’s Rule or derivatives on this exam.)

(a) \( \lim_{t \to 2} \frac{2 - t}{\sqrt{6} - \sqrt{3t}} \)

(b) \( \lim_{x \to 0} x^2 \cos \left( \frac{1}{x^2} \right) \)

(c) \( \lim_{h \to 0} \frac{\sin(\pi + h) - \sin(\pi - h)}{2h} \)  

(Hint: Refer to the formula section.)

Solution:

(a) \( \lim_{t \to 2} \frac{2 - t}{\sqrt{6} - \sqrt{3t}} \cdot \frac{\sqrt{6} + \sqrt{3t}}{\sqrt{6} + \sqrt{3t}} = \lim_{t \to 2} \frac{(2 - t)(\sqrt{6} + \sqrt{3t})}{6 - 3t} = \lim_{t \to 2} \frac{(2 - t)(\sqrt{6} + \sqrt{3t})}{3(2 - t)} = \frac{2\sqrt{6}}{3} \)

(b)

\(-1 \leq \cos \left( \frac{1}{x^2} \right) \leq 1 \)

\(-x^2 \leq \cos x^2 \left( \frac{1}{x^2} \right) \leq x^2 \)

By the Squeeze Theorem, since \( \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0 \), the value of \( \lim_{x \to 0} x^2 \cos x^2 \left( \frac{1}{x^2} \right) \) also is \( 0 \).

(c) By the formulas

\[
\sin(\pi + h) - \sin(\pi - h) = (\sin(\pi) \cos(h) + \cos(\pi) \sin(h)) - (\sin(\pi) \cos(h) - \cos(\pi) \sin(h)) = 2 \cos(\pi) \sin(h),
\]

thus

\[
\lim_{h \to 0} \frac{\sin(\pi + h) - \sin(\pi - h)}{2h} = \lim_{h \to 0} 2 \cos(\pi) \cdot \frac{\sin(h)}{2h} = \lim_{h \to 0} \cos(\pi) \cdot \frac{\sin(h)}{h} = \cos(\pi) = -1
\]
3. (20 pts) Consider the function \( f(x) = \frac{1}{x+1} \).

(a) Sketch a graph of \( y = f(x) \) on \([-3, 3]\). Label all intercepts.

(b) Suppose we use the precise definition of a limit to verify that \( \lim_{x \to 1} f(x) = \frac{1}{2} \).

   i. Illustrate by adding to your graph from part (a). Include labels for \( \delta \) and \( \epsilon \).

   (You may sketch a new graph instead.)

   ii. Given \( \epsilon = \frac{1}{8} \), what is a corresponding value of \( \delta \)?

(c) Find \( (f \circ f)(x) \) and simplify your answer.

(d) What is the domain of \( f \circ f \)? Write your answer in interval notation.

Solution:

(a)

\[
y = \frac{1}{x+1}
\]

(b) i. 

\[
\begin{align*}
\frac{1}{x+1} &= \frac{3}{8} \quad \Rightarrow \quad x = \frac{5}{3} \\
\frac{1}{x+1} &= \frac{5}{8} \quad \Rightarrow \quad x = \frac{3}{5} \\
\frac{3}{8} < \frac{1}{x+1} < \frac{5}{8} &\quad \Rightarrow \quad \frac{3}{5} < x < \frac{5}{3}. \quad \text{Choose } \delta = 1 - \frac{3}{5} = \frac{2}{5}.
\end{align*}
\]

(c) \( (f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x+1}\right) = \frac{1}{\frac{1}{x+1} + 1} = \frac{1}{x+2} = \frac{x+1}{x+2} \).

(d) The domain of \( f \circ f \) is \((-\infty, -2) \cup (-2, -1) \cup (-1, \infty)\).
4. (20 pts) Let \( g(x) = \frac{7x + 21}{3 - |x|} \).

(a) Express \( g(x) \) as a piecewise function without using absolute value.

(b) Is \( g \) continuous at \( x = 0 \)? Justify your answer using the definition of continuity.

(c) Identify any horizontal asymptotes of the curve \( y = g(x) \). Justify your answer using appropriate limits.

Solution:

(a) 
\[
g(x) = \begin{cases} 
\frac{7x + 21}{3 - x} & x \geq 0 \\
\frac{7x + 21}{3 + x} & x < 0
\end{cases}
\]

(b) \( g \) is continuous at \( x = 0 \) if 
\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0).
\]

\[
g(0) = \frac{21}{3} = 7.
\]

\[
\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{7x + 21}{3 - x} = \frac{21}{3} = 7.
\]

\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \frac{7x + 21}{3 + x} = \frac{21}{3} = 7.
\]

So indeed, \( f \) is continuous at 0.

(c) 
\[
\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{7x + 21}{3 - x} = \lim_{x \to \infty} \frac{7 + 21/x}{3/x - 1} = \frac{7}{-1} = -7.
\]

\[
\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{7x + 21}{3 + x} = \lim_{x \to -\infty} \frac{7 + 21/x}{3/x + 1} = \frac{7}{1} = 7.
\]

So \( y = -7 \) and \( y = 7 \) are horizontal asymptotes of the curve \( y = g(x) \).
5. (21 pts) The following problems are not related.

(a) Find and classify all discontinuities of \( h(x) = \frac{2x^2 - 9x + 9}{x^2 - 9} \). Justify using appropriate limits.

(b) Show that the function \( f(x) = \sqrt{x} \cos(x) \) intersects the line \( y = -1 \). Based on your work, indicate an interval where the intersection point can be found.

Solution:

(a) Factoring the top and bottom gives

\[
h(x) = \frac{2x^2 - 9x + 9}{x^2 - 9} = \frac{(2x - 3)(x - 3)}{(x + 3)(x - 3)}
\]

There is a **removable discontinuity** at \( x = 3 \):

\[
\lim_{x \to 3} \frac{(2x - 3)(x - 3)}{(x + 3)(x - 3)} = \frac{3}{6} = \frac{1}{2}
\]

There is an **infinite discontinuity** at \( x = -3 \):

\[
\lim_{x \to -3^+} \frac{2x - 3}{x + 3} = -\infty
\]

since the numerator approaches \(-9\) and the denominator approaches 0 with positive values.

(b) For this question, we need to use the Intermediate Value Theorem, since the question effectively asks us to prove that a root of the equation \( f(x) = \sqrt{x} \cos(x) + 1 \) exists. We need to find a closed interval \([a, b]\) for which \( f(x) \) is continuous and \( f(a) \) has opposite sign to \( f(b) \). Two values that work are \( a = 0 \) and \( b = \pi \). Since trig functions and root functions are continuous on their domains, we know that \( f(x) \) is continuous on \([0, \pi]\). Since \( f(0) = 1 > 0 \) and \( f(\pi) = -\sqrt{\pi} + 1 < 0 \), by IVT, there is some \( c \) in the interval \((0, \pi)\) for which \( f(c) = 0 \).