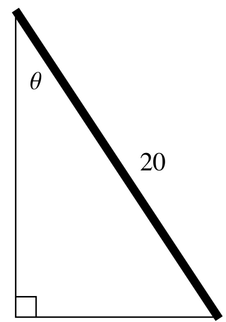
1. (15 pts) The following two problems are not related.

   (a) Solve the inequality on $[0, 2\pi]$ and write your answer in interval notation.

   $$2\cos x \sec x \sin x + 1 < 0$$

   (b) A 20-ft ladder leaning against a wall forms an angle of θ with the wall, with a maximum angle of $\frac{\pi}{3}$ radians. For larger angles the base of the ladder slips and the ladder falls to the ground. What is the slope of the ladder at this maximum angle?



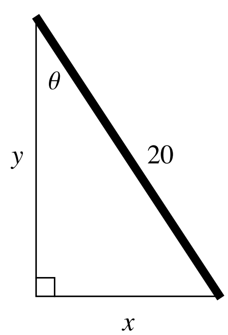
Solution:

(a)

$$2\cos x \sec x \sin x + 1 < 0$$
$$2\sin x + 1 < 0$$
$$\sin x < -\frac{1}{2}$$

The inequality is satisfied when x is in $\boxed{\left(\frac{7\pi}{6}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)}$. Note that $\sec x$ is undefined at $x = \frac{3\pi}{2}$.

(b) The slope of the ladder is $y/x < 0$. When $\theta = \pi/3$ the slope is $-\cot\frac{\pi}{3} = \boxed{-\frac{1}{\sqrt{3}}} = \boxed{-\frac{\sqrt{3}}{3}}$.



2. (24 pts) Evaluate the following limits and simplify your answers.
(*Reminder:* You may not use L'Hopital's Rule or derivatives on this exam.)

(a) $\lim_{t\to 2} \dfrac{2-t}{\sqrt{6}-\sqrt{3t}}$

(b) $\lim_{x\to 0} x^2 \cos\left(\dfrac{1}{x^2}\right)$

(c) $\lim_{h\to 0} \dfrac{\sin(\pi+h)-\sin(\pi-h)}{2h}$ (*Hint:* Refer to the formula section.)

Solution:

(a) $\lim_{t\to 2} \dfrac{2-t}{\sqrt{6}-\sqrt{3t}} \cdot \dfrac{\sqrt{6}+\sqrt{3t}}{\sqrt{6}+\sqrt{3t}} = \lim_{t\to 2} \dfrac{(2-t)\left(\sqrt{6}+\sqrt{3t}\right)}{6-3t} = \lim_{t\to 2} \dfrac{\cancel{(2-t)}\left(\sqrt{6}+\sqrt{3t}\right)}{3\cancel{(2-t)}} = \boxed{\dfrac{2\sqrt{6}}{3}}.$

(b)

$$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$$
$$-x^2 \le \cos x^2 \left(\frac{1}{x^2}\right) \le x^2$$

By the Squeeze Theorem, since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, the value of $\lim_{x\to 0} x^2 \cos x^2 \left(\dfrac{1}{x^2}\right)$ also is $\boxed{0}$.

(c) By the formulas

$$\begin{aligned}\sin(\pi+h)-\sin(\pi-h) &= (\cancel{\sin(\pi)\cos(h)}+\cos(\pi)\sin(h)) - (\cancel{\sin(\pi)\cos(h)}-\cos(\pi)\sin(h)) \\ &= 2\cos(\pi)\sin(h),\end{aligned}$$

thus

$$\lim_{h\to 0} \frac{\sin(\pi+h)-\sin(\pi-h)}{2h} = \lim_{h\to 0} 2\cos(\pi)\frac{\sin(h)}{2h} = \lim_{h\to 0} \cos(\pi)\cdot\frac{\sin(h)}{h} = \cos(\pi) = \boxed{-1}.$$

3. (20 pts) Consider the function $f(x) = \dfrac{1}{x+1}$.

(a) Sketch a graph of $y = f(x)$ on $[-3, 3]$. Label all intercepts.

(b) Suppose we use the precise definition of a limit to verify that $\lim_{x\to1} f(x) = \frac{1}{2}$.

i. Illustrate by adding to your graph from part (a). Include labels for δ and ϵ. (You may sketch a new graph instead.)

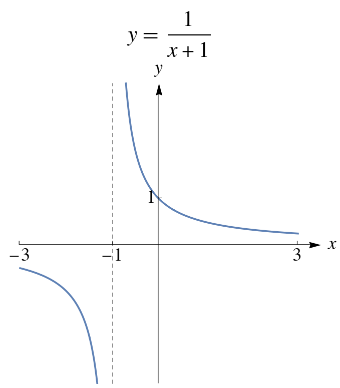
ii. Given $\epsilon = \frac{1}{8}$, what is a corresponding value of δ?

(c) Find $(f \circ f)(x)$ and simplify your answer.

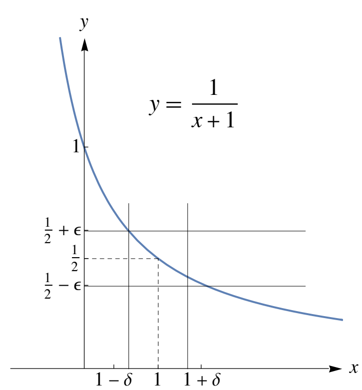
(d) What is the domain of $f \circ f$? Write your answer in interval notation.

Solution:

(a)



(b) i.



ii. $\frac{1}{x+1} = \frac{3}{8} \Rightarrow x = \frac{5}{3}$

$\frac{1}{x+1} = \frac{5}{8} \Rightarrow x = \frac{3}{5}$

$\frac{3}{8} < \frac{1}{x+1} < \frac{5}{8} \Rightarrow \frac{3}{5} < x < \frac{5}{3}$. Choose $\delta = 1 - \frac{3}{5} = \boxed{\frac{2}{5}}$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x+1}\right) = \dfrac{1}{\frac{1}{x+1} + 1} = \dfrac{1}{\frac{x+2}{x+1}} = \boxed{\frac{x+1}{x+2}}$.

(d) The domain of $f \circ f$ is $\boxed{(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)}$.

4. (20 pts) Let $g(x) = \dfrac{7x+21}{3-|x|}$.

(a) Express $g(x)$ as a piecewise function without using absolute value.

(b) Is g continuous at $x = 0$? Justify your answer using the definition of continuity.

(c) Identify any horizontal asymptotes of the curve $y = g(x)$. Justify your answer using appropriate limits.

Solution:

(a)

$$g(x) = \begin{cases} \dfrac{7x+21}{3-x} & x \geq 0 \\ \dfrac{7x+21}{3+x} & x < 0 \end{cases}$$

(b) g is continuous at $x = 0$ if

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0).$$

$$g(0) = \frac{21}{3} = 7.$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{7x+21}{3-x} = \frac{21}{3} = 7.$$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{7x+21}{3+x} = \frac{21}{3} = 7.$$

So indeed, f is continuous at 0.

(c)

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{7x+21}{3-x} = \lim_{x\to\infty} \frac{7+21/x}{3/x-1} = \frac{7}{-1} = -7.$$

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{7x+21}{3+x} = \lim_{x\to-\infty} \frac{7+21/x}{3/x+1} = \frac{7}{1} = 7.$$

So $\boxed{y=-7}$ and $\boxed{y=7}$ are horizontal asymptotes of the curve $y = g(x)$.

5. (21 pts) The following problems are not related.

(a) Find and classify all discontinuities of $h(x) = \dfrac{2x^2 - 9x + 9}{x^2 - 9}$. Justify using appropriate limits.

(b) Show that the function $f(x) = \sqrt{x}\cos x$ intersects the line $y = -1$. Based on your work, indicate an interval where the intersection point can be found.

Solution:

(a) Factoring the top and bottom gives

$$h(x) = \frac{2x^2 - 9x + 9}{x^2 - 9} = \frac{(2x-3)(x-3)}{(x+3)(x-3)}$$

There is a $\boxed{\text{removable discontinuity}}$ at $\boxed{x = 3}$:

$$\lim_{x \to 3} \frac{(2x-3)\cancel{(x-3)}}{(x+3)\cancel{(x-3)}} = \frac{3}{6} = \frac{1}{2}.$$

There is an $\boxed{\text{infinite discontinuity}}$ at $\boxed{x = -3}$:

$$\lim_{x \to -3^+} \frac{2x-3}{x+3} = -\infty$$

since the numerator approaches -9 and the denominator approaches 0 with positive values.

(b) For this question, we need to use the Intermediate Value Theorem, since the question effectively asks us to prove that a root of the equation $f(x) = \sqrt{x}\cos(x) + 1$ exists. We need to find a closed interval $[a, b]$ for which $f(x)$ is continuous and $f(a)$ has opposite sign to $f(b)$. Two values that work are $a = 0$ and $b = \pi$. Since trig functions and root functions are continuous on their domains, we know that $f(x)$ is continuous on $[0, \pi]$. Since $f(0) = 1 > 0$ and $f(\pi) = -\sqrt{\pi} + 1 < 0$, by IVT, there is some c in the interval $\boxed{(0, \pi)}$ for which $f(c) = 0$.