1. (a) (16 pts) Evaluate the following limits.

   i. $\displaystyle\lim_{x\to\pi}\frac{3^{\sin(x)}-1}{x-\pi}$ ii. $\displaystyle\lim_{x\to\infty}\frac{\sinh(x)}{e^x}$

   Solution:

   i. $$\lim_{x\to\pi}\frac{3^{\sin(x)}-1}{x-\pi}\overset{LH}{=}\lim_{x\to\pi}\frac{3^{\sin x}(\ln 3)\cos x}{1}=3^0(\ln 3)(-1)=\boxed{-\ln 3}.$$

   Alternate Solution:

   Use the derivative formula $f'(a)=\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ with $f(x)=3^{\sin x}$ and $a=\pi$.

   $$\lim_{x\to\pi}\frac{3^{\sin(x)}-1}{x-\pi}=\frac{d}{dx}\left(3^{\sin x}\right)\Big|_{x=\pi}=3^{\sin x}(\ln 3)\cos x\Big|_{x=\pi}=\boxed{-\ln 3}.$$

   ii. $$\lim_{x\to\infty}\frac{\sinh(x)}{e^x}=\lim_{x\to\infty}\frac{\frac{1}{2}\left(e^x-e^{-x}\right)}{e^x}=\lim_{x\to\infty}\frac{1}{2}\left(1-e^{-2x}\right)=\frac{1}{2}(1-0)=\boxed{\frac{1}{2}}.$$

   (b) (16 pts) Consider $f(x)=e^{1/x}$.

   i. Find the domain of the function. Express your answer in interval notation.

   ii. Evaluate $\displaystyle\lim_{x\to 0^-}f(x)$ and $\displaystyle\lim_{x\to 0^+}f(x)$.

   iii. Find the horizontal asymptotes of f, if any. Justify using appropriate limits.

   Solution:

   i. $\boxed{(-\infty,0)\cup(0,\infty)}$

   ii. $\displaystyle\lim_{x\to 0^-}e^{1/x}=\boxed{0}$ because $\displaystyle\lim_{x\to 0^-}\frac{1}{x}=-\infty$ and $\displaystyle\lim_{x\to 0^+}e^{1/x}=\boxed{\infty}$ because $\displaystyle\lim_{x\to 0^+}\frac{1}{x}=\infty$.

   iii. $\displaystyle\lim_{x\to\infty}e^{1/x}=e^0=1$ and $\displaystyle\lim_{x\to-\infty}e^{1/x}=e^0=1$. There is a horizontal asymptote at $\boxed{y=1}$.

2. (30 pts) The following problems are not related.

   (a) The curve given by the equation $x^3+y^3=6xy$ is called the *folium of Descartes*. Find an equation of the tangent line to the folium of Descartes at the point $(3,3)$. Write your answer in the form $y=mx+b$.

   (b) Let $g(x)=\displaystyle\int_{4/x}^{1}t\cot(t)\,dt$. Find $\dfrac{dg}{dx}$.

   (c) Given $y=(\ln x)^{\cos x}$, find y' at $x=\pi$. Simplify your answer.

   Solution:

(a)

$$\begin{aligned}
x^3+y^3&=6xy\\
3x^2+3y^2\frac{dy}{dx}&=6x\frac{dy}{dx}+6y\\
\frac{dy}{dx}\left(3y^2-6x\right)&=6y-3x^2\\
\frac{dy}{dx}&=\frac{6y-3x^2}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}\\
\left.\frac{dy}{dx}\right|_{(3,3)}&=\frac{6-9}{9-6}=-1
\end{aligned}$$

The equation of the tangent line is $y=3-(x-3)$ or $\boxed{y=-x+6}$.

(b) Use the FTC and the chain rule.

$$\frac{dg}{dx}=\frac{d}{dx}\int_{4/x}^{1}t\cot(t)\,dt=\frac{d}{dx}\int_{1}^{4/x}-t\cot(t)\,dt=-\frac{4}{x}\cot(4/x)\left(-\frac{4}{x^2}\right)=\boxed{\frac{16}{x^3}\cot(4/x)}.$$

(c)

$$\begin{aligned}
y&=(\ln x)^{\cos x}\\
\ln y&=\ln\left((\ln x)^{\cos x}\right)\\
&=(\cos x)\ln(\ln x)\\
\frac{1}{y}\frac{dy}{dx}&=\frac{\cos x}{x\ln x}-(\sin x)\ln(\ln x)\\
\frac{dy}{dx}&=(\ln x)^{\cos x}\left(\frac{\cos x}{x\ln x}-(\sin x)\ln(\ln x)\right)\\
\left.\frac{dy}{dx}\right|_{x=\pi}&=(\ln\pi)^{-1}\left(\frac{-1}{\pi\ln\pi}-0\right)=\boxed{\frac{-1}{\pi(\ln\pi)^2}}
\end{aligned}$$

3. (12 pts) Two particles are moving along a line. Particle A's position is given by $A(t)=t^2+2+3\sin^2(t)$. Particle B's position is given by $B(t)=t-3\cos^2(t)$. At what time are A and B closest?

Solution:

Let $D(t)$ represent the distance between A and B at time t. Note that $A(0)=2>B(0)=-3$.

$$\begin{aligned}
D(t)&=A(t)-B(t)\\
&=\left(t^2+2+3\sin^2(t)\right)-\left(t-3\cos^2(t)\right)\\
&=t^2-t+2+3\left(\sin^2(t)+\cos^2(t)\right)\\
&=t^2-t+5\\
\frac{dD}{dt}&=2t-1\\
\frac{dD}{dt}&=0\implies t=1/2
\end{aligned}$$

Since $d^2D/dt^2=2>0$, the function $D(t)$ is concave up and has a minimum value at $\boxed{t=1/2}$.

4. (18 pts) The following problems are not related.

(a) The sum $S = 11 + 13 + 15 + \cdots + 99$ of odd integers can be written as $\sum_{i=1}^{n} a_i$. Find the value of n and an expression for a_i.

(b) Approximate the value of $\int_0^1 \cos^{-1}(x)\,dx$ using L_2, the left-endpoint rectangle approximation with two equal subintervals. Simplify your answer.

(c) Suppose f is a twice-differentiable function and the linearization of $y = x^2 - f(x)$ at $x = 5$ is used to approximate values of y near $x = 5$. Given that $f''(5) = -1$, will the approximations be underestimates or overestimates? Explain.

Solution:

(a) Let $n = \boxed{45}$ and $a_i = \boxed{2i + 9}$. Then $S = 11 + 13 + 15 + \cdots + 99 = \sum_{i=1}^{45}(2i + 9)$.

(b) Let $f(x) = \cos^{-1}(x)$ and $\Delta x = 1/2$.

$$\begin{aligned} L_2 &= \Delta x\,(f(0) + f(1/2)) \\ &= \frac{1}{2}\left(\cos^{-1}(0) + \cos^{-1}(1/2)\right) = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \boxed{\frac{5\pi}{12}}. \end{aligned}$$

(c) The derivatives of y are $y' = 2x - f'(x)$ and $y'' = 2 - f''(x)$. Since $y''(5) = 2 - (-1) = 3$, the function y is concave up at $x = 5$ so the tangent line lies below the curve, resulting in approximations that are $\boxed{\text{underestimates}}$.

5. (24 pts) Evaluate the following integrals.

(a) $\int_0^{10} h(x)\,dx$ given $h(x) = \begin{cases} |x-2| & x \le 6 \\ 4 & x > 6 \end{cases}$.

(b) $\int \frac{1-t}{\sqrt{9 + 2t - t^2}}\,dt$

(c) $\int \frac{\sinh x}{1 + \cosh^2 x}dx$

Solution:

(a) The area under the curve corresponds to the area of two right triangles and a rectangle.

$$\int_0^{10} h(x)\,dx = \frac{1}{2}(2)(2) + \frac{1}{2}(4)(4) + (4)(4) = 2 + 8 + 16 = \boxed{26}.$$

Alternate solution:

$$\begin{aligned} \int_0^{10} h(x)\,dx &= \int_0^2 (-x + 2)\,dx + \int_2^6 (x - 2)\,dx + \int_6^{10} 4\,dx \\ &= \left[-\frac{x^2}{2} + 2x\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^6 + [4x]_6^{10} \\ &= [-2 + 4 - 0] + [18 - 12 - (2 - 4)] + [40 - 24] \\ &= 2 + 8 + 16 = \boxed{26}. \end{aligned}$$

(b) Let $u = 9 + 2t - t^2$, $du = (2 - 2t)\,dt = 2(1 - t)\,dt$.

$$\int \frac{1-t}{\sqrt{9+2t-t^2}}\,dt = \int \frac{1}{2}\frac{du}{\sqrt{u}} = \int \frac{1}{2}u^{-1/2}\,du = \frac{1}{2}\left(2u^{1/2}\right) + C = \boxed{\sqrt{9+2t-t^2}+C}.$$

(c) Let $u = \cosh x$, $du = \sinh x\,dx$.

$$\int \frac{\sinh x}{1+\cosh^2 x}dx = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \boxed{\tan^{-1}(\cosh x) + C}.$$

6. (12 pts) A radioactive substance decreases in mass by 17% in 3 years. What is the half-life of the substance?

Solution:

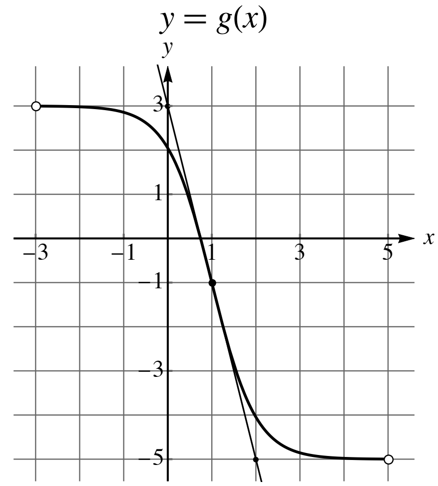
Let $m(t)$ equal the mass of the substance after t years. We are given that $m(3) = 0.83m(0)$. First find the rate constant k.

$$\begin{aligned} m(t) &= m(0)e^{kt} \\ m(3) &= m(0)e^{3k} = 0.83m(0) \\ e^{3k} &= 0.83 \\ 3k &= \ln(0.83) \\ k &= \frac{\ln(0.83)}{3} \end{aligned}$$

Now find the half-life by solving $m(t) = \frac{1}{2}m(0)$ for t.

$$\begin{aligned} m(t) &= m(0)e^{kt} = \frac{1}{2}m(0) \\ e^{kt} &= \frac{1}{2} \\ kt &= \ln(1/2) \\ t &= \frac{\ln(1/2)}{k} = \boxed{\frac{3\ln(1/2)}{\ln(0.83)}\text{ yrs}} = \boxed{\frac{-3\ln 2}{\ln(0.83)}\text{ yrs}} \approx 11.2\text{ yrs} \end{aligned}$$

7. (22 pts)



Shown above is the graph of $y = g(x)$ defined on $(-3, 5)$, and the line tangent to g at $(1, -1)$.

(a) Find the value of $\lim_{x \to 0} |g(x) - 5|$.

(b) Find the value of $\lim_{h \to 0} \dfrac{g(1+h) - g(1)}{h}$.

(c) Suppose g is the derivative of a function f so that $g(x) = f'(x)$. Estimate the interval(s) on which f is increasing, or write NONE if f is a decreasing function.

(d) Find the following values for the inverse function of g.

i. $g^{-1}(2)$ 　　　　 ii. $\left(g^{-1}\right)'(-1)$

Solution:

(a) $\lim_{x \to 0} |g(x) - 5| = |g(0) - 5| = |2 - 5| = \boxed{3}$.

(b) The limit corresponds to the slope of the tangent line through the points $(0, 3)$ and $(1, -1)$.
$\lim_{h \to 0} \dfrac{g(1+h) - g(1)}{h} = g'(1) = \dfrac{-1-3}{1-0} = \boxed{-4}$.

(c) f is increasing where $f' = g$ is positive on $\boxed{(-3, 0.74)}$ or similar.
(Note that the curve lies to the left of the tangent line which has an x-intercept at 0.75.)

(d) i. $g^{-1}(2) = \boxed{0}$.
ii. Note that $g^{-1}(-1) = 1$. Then $\left(g^{-1}\right)'(-1) = 1/g'(1) = 1/(-4) = \boxed{-1/4}$.