1. **Core Section: Integration** (30 pts)

   (a) $\displaystyle\int \frac{\csc^2(1/t)}{t^2}\,dt$

   (b) $\displaystyle\int_0^4 \left(|x-1| + \sqrt{16-x^2}\right) dx$

   (c) Find the average value of $f(x) = \dfrac{18x}{(x^2+9)^2}$ on $[0,3]$.

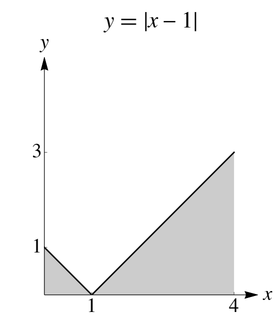
   (d) Find the value of $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(4 - 5\left(-2 + \frac{2i}{n}\right)^4\right)\frac{2}{n}$. (*Hint: Evaluate an integral.*)

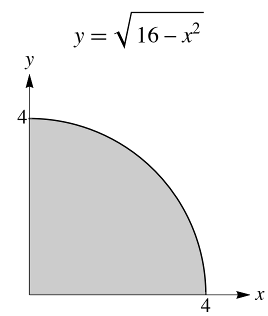
Solution:

(a) (7 pt) Let $u = 1/t,\ \ du = -1/t^2\,dt$.

$$\int \frac{\csc^2(1/t)}{t^2}\,dt = \int -\csc^2 u\,du = \cot u + C = \boxed{\cot(1/t) + C}.$$

(b) (8 pt) $\displaystyle\int_0^4 \left(|x-1| + \sqrt{16-x^2}\right) dx = \frac{1}{2}(1) + \frac{1}{2}(3)(3) + \frac{1}{4}(16\pi) = \boxed{4\pi + 5}.$





(c) (8 pt) Let $u = x^2 + 9,\ du = 2x\,dx$.

$$f_{ave} = \frac{1}{3}\int_0^3 \frac{18x}{(x^2+9)^2}\,dx = \frac{1}{3}\int_9^{18} 9u^{-2}\,du = \frac{1}{3}\left[-\frac{9}{u}\right]_9^{18} = 3\left(-\frac{1}{18} + \frac{1}{9}\right) = \boxed{\frac{1}{6}}.$$

(d) (7 pt) The expression corresponds to

$$\int_{-2}^{0} \left(4 - 5x^4\right) dx = \left[4x - x^5\right]_{-2}^{0} = 0 - (-8 + 32) = \boxed{-24}$$

$$\text{or } \int_0^2 \left(4 - 5(-2+x)^4\right) dx = \boxed{-24}.$$

2. (15 pts) Beulah Bug is crawling along the curve $y = x^2$. Find the x-coordinates of the points on the curve where Beulah is closest to her home located at $(0, 1)$.

Solution: Let D be the distance from the point $(0, 1)$ to a point (x, x^2) on the curve. We can minimize $S = D^2$ to find the critical numbers.

$$D = \sqrt{x^2 + (x^2 - 1)^2}$$
$$S = D^2 = x^2 + \left(x^2 - 1\right)^2 = x^4 - x^2 + 1$$

Find the critical numbers.

$$S' = 4x^3 - 2x$$
$$S' = 0 \Rightarrow 2x(2x^2 - 1) = 0 \Rightarrow x = 0, \pm\frac{1}{\sqrt{2}}.$$

Now perform the second derivative test.

$$S'' = 12x^2 - 2, \quad S''(0) < 0, \quad S''\left(\pm\frac{1}{\sqrt{2}}\right) > 0.$$

There is a local maximum at $x = 0$ and there are minima at $x = \boxed{\pm 1/\sqrt{2}}$, where Beulah is closest to her home.

3. (24 pts) The following three problems are not related.

(a) Find an equation for the slant asymptote of $y = \dfrac{9x^2 + 3x}{3x + 5}$. (It is not necessary to justify your answer with limits.)

(b) If $\displaystyle\sum_{i=1}^{n}(4i - 2) = 7200$, what is the value of n?

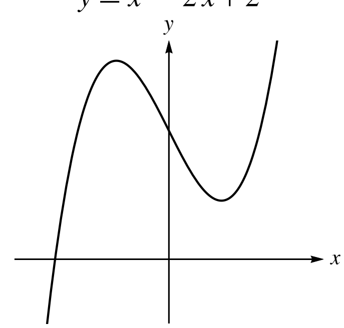
(c) Apply Newton's Method to $y = x^3 - 2x + 2$ with an initial approximation of $x_1 = 0$.

i. Calculate the next two approximations x_2 and x_3.

ii. Copy the graph of the function. (It is not necessary to draw a precise graph.) Sketch the lines used to find x_2 and x_3.

iii. Explain what happens to x_n as $n \to \infty$. Will x_n approach the root?

$y = x^3 - 2x + 2$



Solution:

(a) (7 pt)

$$\begin{array}{r|l} & 3x - 4 \\ \hline 3x+5 & 9x^2 + 3x \\ & -9x^2 - 15x \\ \hline & -12x \\ & 12x + 20 \\ \hline & 20 \end{array}$$

There is a slant asymptote at $\boxed{y = 3x - 4}$.

(b) (7 pt) $\sum_{i=1}^{n}(4i-2) = 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 2 = 4 \cdot \frac{n(n+1)}{2} - 2n = 2\left(n^2+n\right) - 2n = 2n^2.$

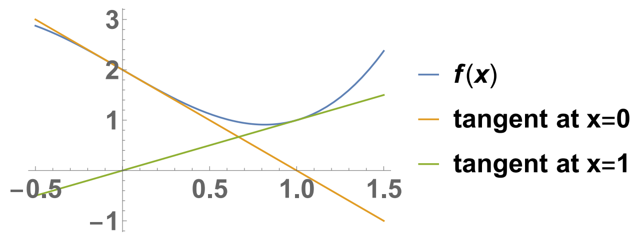
Solving $2n^2 = 7200$ produces $n = \boxed{60}$.

(c) (10 pt)

i. $f(x) = x^3 - 2x + 2$ and $f'(x) = 3x^2 - 2$. Newton's method provides the following values for $x_{n+1} = x_n - f(x_n)/f'(x_n)$.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
1	0	2	-2	$x_2 = \boxed{1}$
2	1	1	1	$x_3 = \boxed{0}$

ii. The following graph shows Newton's method and the two tangent lines.



iii. As $n \to \infty$, the values for x_n cycle between 0 and 1 and will not approach the root.

4. (15 pts) Ralphie is driving her car at 60 feet per second when she sees a red light ahead, 180 feet away. She immediately brakes with a constant acceleration of $-k$ ft/sec^2, $k > 0$.

(a) Find the car's velocity and position functions, $v(t)$ and $s(t)$, in terms of k. Assume that the initial position is 0 feet.

(b) How many seconds will it take for Ralphie to come to a complete stop? Express your answer in terms of k.

(c) Find the value of k that will cause Ralphie to stop right at the light.

Solution:

(a)

$$
\begin{aligned}
a(t) &= -k \\
v(t) &= -kt + C & v(0) = 60 \implies \\
v(t) &= \boxed{-kt + 60} \\
s(t) &= -\frac{k}{2}t^2 + 60t + C & s(0) = 0 \implies \\
s(t) &= \boxed{-\frac{k}{2}t^2 + 60t}
\end{aligned}
$$

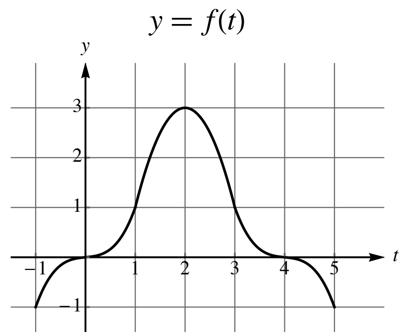
(b) Let t_f be the stopping time in seconds. Then

$$v(t_f) = -kt + 60 = 0 \implies t_f = \boxed{\frac{60}{k}}.$$

(c) Find the appropriate acceleration so that $s(t_f) = 180$.

$$s(t_f) = -\frac{k}{2}\left(\frac{60}{k}\right)^2 + 60\left(\frac{60}{k}\right) = \frac{1800}{k} = 180 \implies k = \boxed{10}.$$

5. (16 pts)



The graph of a differentiable function f on the interval $[-1, 5]$ is shown above. The graph of f has a horizontal tangent line at $t = 2$. Let $g(x) = 5 + \int_2^x f(t)\,dt$ for $-1 \le x \le 5$. No explanations are necessary for the following questions.

(a) Find the values of $g(2), g'(2)$, and $g''(2)$.

(b) On what interval(s) is g decreasing?

(c) On what interval(s) is g concave down?

Solution:

By the Fundamental Theorem of Calculus, $g'(x) = f(x)$ and $g''(x) = f'(x)$.

(a)

$$g(2) = 5 + \int_2^2 f(t)\,dt = \boxed{5}$$

$$g'(2) = f(2) = \boxed{3}$$

$$g''(2) = f'(2) = \boxed{0}$$

(b) $\boxed{(-1, 0)}$ and $\boxed{(4, 5)}$

(c) $\boxed{(2, 5)}$