1. Core Section: Integration (30 pts)

(a) \[ \int \frac{\csc^2(1/t)}{t^2} \, dt \]

(b) \[ \int_0^4 \left( |x - 1| + \sqrt{16 - x^2} \right) \, dx \]

(c) Find the average value of \( f(x) = \frac{18x}{(x^2 + 9)^2} \) on \([0, 3]\).

(d) Find the value of \( \lim_{n \to \infty} \sum_{i=1}^{n} \left( 4 - 5 \left( -2 + \frac{2i}{n} \right)^4 \right) \frac{2}{n} \). (Hint: Evaluate an integral.)

Solution:

(a) (7 pt) Let \( u = 1/t, \ du = -1/t^2 \, dt \).

\[ \int \frac{\csc^2(1/t)}{t^2} \, dt = \int -\csc^2 u \, du = \cot u + C = \cot(1/t) + C. \]

(b) (8 pt) \[ \int_0^4 \left( |x - 1| + \sqrt{16 - x^2} \right) \, dx = \frac{1}{2} (1) + \frac{1}{2} (3)(3) + \frac{1}{4} (16\pi) = 4\pi + 5. \]

(c) (8 pt) Let \( u = x^2 + 9, \ du = 2x \, dx \).

\[ f_{\text{ave}} = \frac{1}{3} \int_0^3 \frac{18x}{(x^2 + 9)^2} \, dx = \frac{1}{3} \int_9^{18} 9u^{-2} \, du = \frac{1}{3} \left[ -\frac{9}{u} \right]_9^{18} = 3 \left( -\frac{1}{18} + \frac{1}{9} \right) = \frac{1}{6}. \]

(d) (7 pt) The expression corresponds to

\[ \int_{-2}^0 \left( 4 - 5x^4 \right) \, dx = [4x - x^5]_{-2}^0 = 0 - (-8 + 32) = -24 \]

or \[ \int_0^2 \left( 4 - 5(-2 + x)^4 \right) \, dx = -24. \]
2. (15 pts) Beulah Bug is crawling along the curve \( y = x^2 \). Find the \( x \)-coordinates of the points on the curve where Beulah is closest to her home located at \((0, 1)\).

**Solution:** Let \( D \) be the distance from the point \((0, 1)\) to a point \((x, x^2)\) on the curve. We can minimize \( S = D^2 \) to find the critical numbers.

\[
D = \sqrt{x^2 + (x^2 - 1)^2}
\]

\[
S = D^2 = x^2 + (x^2 - 1)^2 = x^4 - x^2 + 1
\]

Find the critical numbers.

\[
S' = 4x^3 - 2x
\]

\[
S' = 0 \Rightarrow 2x(2x^2 - 1) = 0 \Rightarrow x = 0, \pm \frac{1}{\sqrt{2}}.
\]

Now perform the second derivative test.

\[
S'' = 12x^2 - 2, \quad S''(0) < 0, \quad S''\left(\pm \frac{1}{\sqrt{2}}\right) > 0.
\]

There is a local maximum at \( x = 0 \) and there are minima at \( x = \pm \frac{1}{\sqrt{2}} \), where Beulah is closest to her home.

3. (24 pts) The following three problems are not related.

(a) Find an equation for the slant asymptote of \( y = \frac{9x^2 + 3x}{3x + 5} \). (It is not necessary to justify your answer with limits.)

(b) If \( \sum_{i=1}^{n} (4i - 2) = 7200 \), what is the value of \( n \)?

(c) Apply Newton’s Method to \( y = x^3 - 2x + 2 \) with an initial approximation of \( x_1 = 0 \).

i. Calculate the next two approximations \( x_2 \) and \( x_3 \).

ii. Copy the graph of the function. (It is not necessary to draw a precise graph.) Sketch the lines used to find \( x_2 \) and \( x_3 \).

iii. Explain what happens to \( x_n \) as \( n \to \infty \). Will \( x_n \) approach the root?

**Solution:**

(a) (7 pt)

\[
\begin{align*}
\frac{3x - 4}{3x + 5} \\
\frac{9x^2 + 3x}{-9x^2 - 15x} \\
\frac{-12x}{12x + 20} \\
\frac{20}{20}
\end{align*}
\]

There is a slant asymptote at \( y = 3x - 4 \).
(b) \[ \sum_{i=1}^{n} (4i - 2) = 4 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 2 = 4 \cdot \frac{n(n + 1)}{2} - 2n = 2 (n^2 + n) - 2n = 2n^2. \]

Solving \( 2n^2 = 7200 \) produces \( n = 60 \).

(c) (10 pt)

i. \( f(x) = x^3 - 2x + 2 \) and \( f'(x) = 3x^2 - 2 \). Newton’s method provides the following values for \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( f(x_n) )</th>
<th>( f'(x_n) )</th>
<th>( x_{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>( x_2 = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( x_3 = 0 )</td>
</tr>
</tbody>
</table>

ii. The following graph shows Newton’s method and the two tangent lines.

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iii. As \( n \to \infty \), the values for \( x_n \) cycle between 0 and 1 and will not approach the root.

4. (15 pts) Ralphie is driving her car at 60 feet per second when she sees a red light ahead, 180 feet away. She immediately brakes with a constant acceleration of \( -k \text{ ft/sec}^2 \), \( k > 0 \).

(a) Find the car’s velocity and position functions, \( v(t) \) and \( s(t) \), in terms of \( k \). Assume that the initial position is 0 feet.

(b) How many seconds will it take for Ralphie to come to a complete stop? Express your answer in terms of \( k \).

(c) Find the value of \( k \) that will cause Ralphie to stop right at the light.

**Solution:**

(a)

\[
\begin{align*}
da(t) &= -k \\
v(t) &= -kt + C \\
v(0) &= 60 \implies v(t) = -kt + 60 \\
s(t) &= -\frac{k}{2}t^2 + 60t + C \\
s(0) &= 0 \implies s(t) = -\frac{k}{2}t^2 + 60t \\
\end{align*}
\]

(b) Let \( t_f \) be the stopping time in seconds. Then

\[
v(t_f) = -kt + 60 = 0 \implies t_f = \frac{60}{k}.
\]
(c) Find the appropriate acceleration so that \( s(t_f) = 180 \).

\[
s(t_f) = -\frac{k}{2} \left( \frac{60}{k} \right)^2 + 60 \left( \frac{60}{k} \right) = \frac{1800}{k} = 180 \implies k = 10.
\]

5. (16 pts)

The graph of a differentiable function \( f \) on the interval \([-1, 5]\) is shown above. The graph of \( f \) has a horizontal tangent line at \( t = 2 \). Let \( g(x) = 5 + \int_2^x f(t) \, dt \) for \(-1 \leq x \leq 5\). No explanations are necessary for the following questions.

(a) Find the values of \( g(2), g'(2), \) and \( g''(2) \).

(b) On what interval(s) is \( g \) decreasing?

(c) On what interval(s) is \( g \) concave down?

**Solution:**

By the Fundamental Theorem of Calculus, \( g'(x) = f(x) \) and \( g''(x) = f'(x) \).

(a) \[ g(2) = 5 + \int_2^2 f(t) \, dt = 5 \]

\[ g'(2) = f(2) = 3 \]

\[ g''(2) = f'(2) = 0 \]

(b) \((-1, 0)\) and \((4, 5)\)

(c) \((2, 5)\)