1. **Core Section: Differentiation** (30 pts)

   (a) Let \( g(t) = \left( at + \sqrt{a^2 + t^2} \right)^{-2} \). Find \( \frac{dg}{dt} \) and leave your answer unsimplified.

   (b) Given \( 5xy^3 - 2 \sqrt{x^3} = 8y + \sqrt{3} \), find \( \frac{dy}{dx} \). (Similar to WebAssign 2.6 #5)

   (c) Find an equation for the line tangent to \( y = \frac{\sec(3x)}{2 + \tan(3x)} \) at \( x = 0 \).

**Solution:**

(a) \[ g'(t) = -2 \left( at + \sqrt{a^2 + t^2} \right)^{-3} \left( a + \frac{1}{2} \left( a^2 + t^2 \right)^{-1/2} \right) \]

(b) \[ 5xy^3 - 2 \sqrt{x^3} = 8y + \sqrt{3} \]

\[ 5x (3y^2) \frac{dy}{dx} + 5y^3 - 3 \sqrt{x} = 8 \frac{dy}{dx} \]

\[ \frac{dy}{dx} \left( 15xy^2 - 8 \right) = 3 \sqrt{x} - 5y^3 \]

\[ \frac{dy}{dx} = \frac{3 \sqrt{x} - 5y^3}{15xy^2 - 8} \]

(c) \[ y' = \frac{(2 + \tan(3x)) (3 \sec(3x) \tan(3x)) - \sec(3x) (3 \sec^2(3x))}{(2 + \tan(3x))^2} \]

\[ y'(0) = 0 - \frac{3}{2^2} = -\frac{3}{4} \quad \text{and} \quad y(0) = \frac{1}{2} \]

An equation for the tangent line is \( y = \frac{1}{2} - \frac{3}{4}x \).

2. (15 pts) Let \( f(x) = \begin{cases} \frac{8}{3}x & x < 6 \\ (x - 2)^2 & x \geq 6 \end{cases} \)

   (a) Write the limit definition of \( f'(a) \), the derivative of \( f(x) \) at \( x = a \).

   (b) Use the definition to determine the value of \( f'(6) \) or to show that it does not exist. (Similar to HW 4 #4b)

**Solution:**

(a) \[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad \text{or} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]
(b) From the left:
\[
\lim_{h \to 0^-} \frac{\frac{8}{3}(6 + h) - 16}{h} = \lim_{h \to 0^-} \frac{16 + \frac{8}{3}h - 16}{h} = \frac{8}{3}.
\]

From the right:
\[
\lim_{h \to 0^+} \frac{(6 + h - 2)^2 - 16}{h} = \lim_{h \to 0^+} \frac{(h + 4)^2 - 16}{h} = \lim_{h \to 0^+} \frac{h^2 + 8h}{h} = 8.
\]

Since the limit from the left does not equal the limit from the right, \( f'(6) \) does not exist.

3. (15 pts) A street light is mounted at the top of a 17-ft-tall pole. A woman 5 ft tall walks away from the pole with a speed of 4 ft/s along a straight path. How fast is the tip of her shadow moving when she is 34 ft from the pole? (Similar to HW 6 #3)

Solution:

Given: \( \frac{dx}{dt} = 4 \) ft/sec. Find \( \frac{dy}{dt} \) when \( x = 34 \) ft.

By similar triangles,
\[
\frac{17}{5} = \frac{y}{y-x} \Rightarrow 17y - 17x = 5y \Rightarrow 12y = 17x
\]
\[
12 \frac{dy}{dt} = 17 \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{17}{12} \times (4) = \frac{17}{3}.
\]

The tip of the shadow is moving at \( \frac{17}{3} \) ft/sec.

(Note that the answer is independent of the woman’s distance from the pole.)

4. (20 pts) Consider \( f(x) = \frac{x^2 + x + 4}{x + 1} \) with \( f'(x) = 1 - \frac{4}{(x + 1)^2} \) and \( f''(x) = \frac{8}{(x + 1)^3} \). (Similar to WebAssign 3.3 #10)

(a) On what intervals is \( f \) increasing? decreasing?
(b) Find the \( x \) and \( y \) coordinates of the local maximum and minimum values of \( f \).
(c) On what intervals is \( f \) concave up? concave down?
(d) Use parts (a) to (c) to sketch a graph of \( f \).

Solution:

(a) \( f'(x) = 0 \Rightarrow x + 1 = \pm 2 \Rightarrow x = -3, 1 \). \( f' \) is undefined at \( x = -1 \).
The intervals for the function $y$ are:

<table>
<thead>
<tr>
<th>Intervals</th>
<th>$y'$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -3$</td>
<td>+</td>
<td>increasing on $(-\infty, -3)$</td>
</tr>
<tr>
<td>$-3 &lt; x &lt; -1$</td>
<td>-</td>
<td>decreasing on $(-3, -1)$</td>
</tr>
<tr>
<td>$-1 &lt; x &lt; 1$</td>
<td>-</td>
<td>decreasing on $(-1, 1)$</td>
</tr>
<tr>
<td>$x &gt; 1$</td>
<td>+</td>
<td>increasing on $(1, \infty)$</td>
</tr>
</tbody>
</table>

(b) There is a local maximum at $(-3, -5)$ and a local minimum at $(1, 3)$.

(c) $y''$ does not equal 0 for any $x$. It is undefined at $x = -1$.

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<tbody>
<tr>
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<td>-</td>
<td>concave down on $(-\infty, -1)$</td>
</tr>
<tr>
<td>$x &gt; -1$</td>
<td>+</td>
<td>concave up on $(-1, \infty)$</td>
</tr>
</tbody>
</table>

(d) The function $y = \frac{x^2 + x + 4}{x + 1}$ is given.

5. (20 pts)

Shown above is the graph of $y = g(x)$ and the line tangent to $g$ at $(1, -2)$. The function $g$ is differentiable on $(-4, 6) \cup (6, 8)$.

(a) Sketch the graph of $y = g'(x)$. Label tick marks clearly. (Similar to HW 4 #3)

(b) Use the linearization of $g$ at $a = 1$ to estimate the value of $g(0.7)$. (Similar to Exam 2 #5 fall 2017)

(c) The Mean Value Theorem states that there exists a value of $c$ in $(-4, 6)$ such that $g'(c)$ equals a certain value.

i. What is that value?
ii. Suppose we wish to narrow down the possible values for $c$. Between which two consecutive integers can $c$ be found? List all possible answers. No explanation is necessary.

**Solution:**

(a) 

(b) $L(x) = g(1) + g'(1)(x - 1) = -2 + 4(x - 1) \Rightarrow L(0.7) = -2 + 4(0.7 - 1) = -3.2$.

(c) i. $g'(c) = g(6) - g(-4) = 2 - (-6) \Rightarrow g'(c) = \frac{4}{5}$.

ii. The tangent slope is $4/5$ in the intervals $(-1, 0)$ and $(2, 3)$. 

\[ y = g'(x) \]

\[ \begin{array}{c|c|c|c}
-4 & -2 & 2 & 4 & 8 & 8 \\
\hline
-3 & -2 & 4 & \\
\end{array} \]