1. **Core Section: Differentiation** (30 pts)

   (a) Let $g(t) = \left(at + \sqrt{a^2 + t^2}\right)^{-2}$. Find $\dfrac{dg}{dt}$ and leave your answer unsimplified.

   (b) Given $5xy^3 - 2\sqrt{x^3} = 8y + \sqrt{3}$, find dy/dx. *(Similar to WebAssign 2.6 #5)*

   (c) Find an equation for the line tangent to $y = \dfrac{\sec(3x)}{2 + \tan(3x)}$ at $x = 0$.

Solution:

(a) $g'(t) = \boxed{-2\left(at + \sqrt{a^2 + t^2}\right)^{-3}\left(a + \frac{1}{2}\left(a^2 + t^2\right)^{-1/2}(2t)\right)}$.

(b)

$$\begin{aligned} 5xy^3 - 2\sqrt{x^3} &= 8y + \sqrt{3} \\ 5x\left(3y^2\right)\frac{dy}{dx} + 5y^3 - 3\sqrt{x} &= 8\frac{dy}{dx} \\ \frac{dy}{dx}\left(15xy^2 - 8\right) &= 3\sqrt{x} - 5y^3 \\ \frac{dy}{dx} &= \boxed{\frac{3\sqrt{x} - 5y^3}{15xy^2 - 8}}. \end{aligned}$$

(c)

$$y' = \frac{(2 + \tan(3x))\,(3\sec(3x)\tan(3x)) - \sec(3x)\left(3\sec^2(3x)\right)}{(2 + \tan(3x))^2}$$

$$y'(0) = \frac{0 - 3}{2^2} = -\frac{3}{4} \quad \text{and} \quad y(0) = \frac{1}{2}$$

An equation for the tangent line is $\boxed{y = \frac{1}{2} - \frac{3}{4}x}$.

2. (15 pts) Let $f(x) = \begin{cases} \frac{8}{3}x & x < 6 \\ (x-2)^2 & x \geq 6 \end{cases}$.

   (a) Write the limit definition of $f'(a)$, the derivative of $f(x)$ at $x = a$.

   (b) Use the definition to determine the value of $f'(6)$ or to show that it does not exist. *(Similar to HW 4 #4b)*

Solution:

(a) $f'(a) = \boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$ or $\boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$.

(b) From the left:

$$\lim_{h\to 0^-} \frac{\frac{8}{3}(6+h)-16}{h} = \lim_{h\to 0} \frac{16+\frac{8}{3}h-16}{h} = \frac{8}{3}.$$

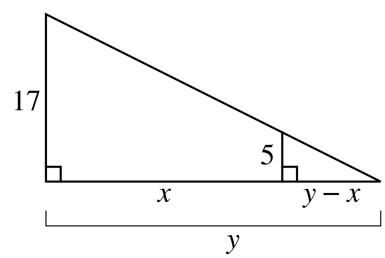
From the right:

$$\lim_{h\to 0^+} \frac{(6+h-2)^2-16}{h} = \lim_{h\to 0^+} \frac{(h+4)^2-16}{h} = \lim_{h\to 0^+} \frac{h^2+8h}{h} = 8.$$

Since the limit from the left does not equal the limit from the right, $f'(6)$ $\boxed{\text{does not exist}}$.

3. (15 pts) A street light is mounted at the top of a 17-ft-tall pole. A woman 5 ft tall walks away from the pole with a speed of 4 ft/s along a straight path. How fast is the tip of her shadow moving when she is 34 ft from the pole? *(Similar to HW 6 #3)*

Solution:



Given: $dx/dt = 4$ ft/sec. Find dy/dt when $x = 34$ ft.

By similar triangles,

$$\frac{17}{5} = \frac{y}{y-x} \Rightarrow 17y - 17x = 5y \Rightarrow 12y = 17x$$

$$12\frac{dy}{dt} = 17\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{17}{12}(4) = \frac{17}{3}.$$

The tip of the shadow is moving at $\boxed{\frac{17}{3}\text{ ft/sec}}$.

(Note that the answer is independent of the woman's distance from the pole.)

4. (20 pts) Consider $f(x) = \dfrac{x^2+x+4}{x+1}$ with $f'(x) = 1 - \dfrac{4}{(x+1)^2}$ and $f''(x) = \dfrac{8}{(x+1)^3}$. *(Similar to WebAssign 3.3 #10)*

(a) On what intervals is f increasing? decreasing?

(b) Find the x and y coordinates of the local maximum and minimum values of f.

(c) On what intervals is f concave up? concave down?

(d) Use parts (a) to (c) to sketch a graph of f.

Solution:

(a) $f'(x) = 0 \Rightarrow x+1 = \pm 2 \Rightarrow x = -3, 1$. f' is undefined at $x = -1$.

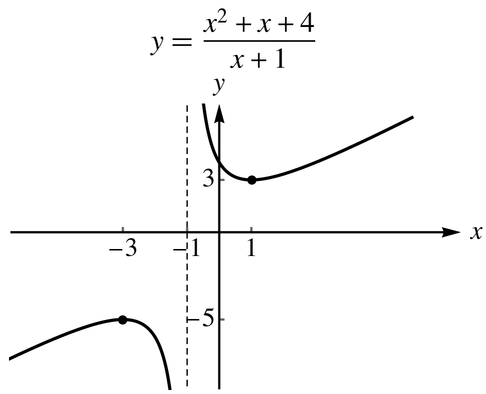
Intervals	y'	y
$x < -3$	+	increasing on $(-\infty, -3)$
$-3 < x < -1$	–	decreasing on $(-3, -1)$
$-1 < x < 1$	–	decreasing on $(-1, 1)$
$x > 1$	+	increasing on $(1, \infty)$

(b) There is a local maximum at $\boxed{(-3, -5)}$ and a local minimum at $\boxed{(1, 3)}$.

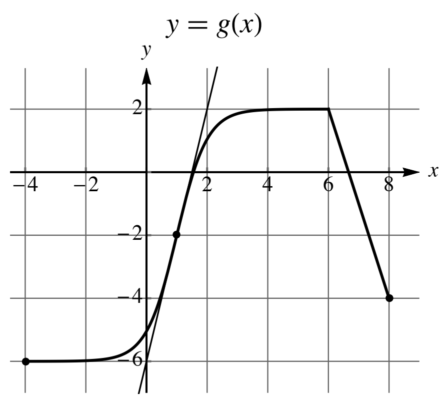
(c) y'' does not equal 0 for any x. It is undefined at $x = -1$.

Intervals	y''	y
$x < -1$	–	concave down on $(-\infty, -1)$
$x > -1$	+	concave up on $(-1, \infty)$

(d)



5. (20 pts)



Shown above is the graph of $y = g(x)$ and the line tangent to g at $(1, -2)$. The function g is differentiable on $(-4, 6) \cup (6, 8)$.

(a) Sketch the graph of $y = g'(x)$. Label tick marks clearly. *(Similar to HW 4 #3)*

(b) Use the linearization of g at $a = 1$ to estimate the value of $g(0.7)$. *(Similar to Exam 2 #5 fall 2017)*

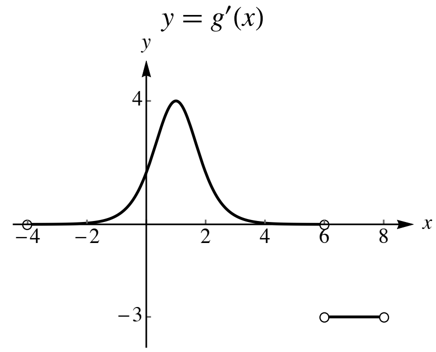
(c) The Mean Value Theorem states that there exists a value of c in $(-4, 6)$ such that $g'(c)$ equals a certain value.

i. What is that value?

ii. Suppose we wish to narrow down the possible values for c. Between which two consecutive integers can c be found? List all possible answers. No explanation is necessary.

Solution:

(a)



(b) $L(x) = g(1) + g'(1)(x-1) = -2 + 4(x-1) \Rightarrow L(0.7) = -2 + 4(0.7-1) = \boxed{-3.2}$.

(c) i. $g'(c) = \dfrac{g(6) - g(-4)}{6 - (-4)} = \dfrac{2 - (-6)}{10} = \boxed{\dfrac{4}{5}}$.

ii. The tangent slope is $4/5$ in the intervals $\boxed{(-1, 0) \text{ and } (2, 3)}$.