APPM 1345
Exam 3
Spring 2024

Name	
Instructor Richard McNamara	**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

Formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

1. (23 pts) Parts (a) and (b) are unrelated.

   (a) Find the inverse function of $f(x) = \dfrac{\ln(2x)}{1+\ln(2x)}$ for $x \geq \dfrac{1}{2}$.

   Express your answer in the form $f^{-1}(x)$. (You do not have to identify the inverse function's domain.)

   Solution:

$$
\begin{aligned}
y &= \frac{\ln(2x)}{1+\ln(2x)} \\
y[1+\ln(2x)] &= \ln(2x) \\
y + y\ln(2x) &= \ln(2x) \\
(y-1)\ln(2x) &= -y \\
\ln(2x) &= \frac{y}{1-y} \\
2x &= e^{y/(1-y)} \\
x &= \frac{1}{2}\, e^{y/(1-y)}
\end{aligned}
$$

   Reverse the roles of x and y to get $\boxed{y = f^{-1}(x) = \frac{1}{2}\, e^{x/(1-x)}}$

(b) Consider the function $g(x) = 2x - \cos x$.

i. Explain why g is invertible, based on its derivative.

ii. Find an equation of the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(4\pi - 1, 2\pi)$.
Hint: Do not attempt to identify the function $g^{-1}(x)$.

Solution:

i. $g'(x) = 2 + \sin x$, which is positive for all real numbers x since $-1 \leq \sin x \leq 1$.

Therefore, $g(x)$ is a monotone increasing function, which implies that it is invertible.

ii. The slope of the line that is tangent to the curve $y = g^{-1}(x)$ at the point $(4\pi - 1, 2\pi)$ is $(g^{-1})'(4\pi - 1)$.

Since $(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$, we know that $(g^{-1})'(4\pi - 1) = \dfrac{1}{g'(g^{-1}(4\pi - 1))}$.

Since the curve $y = g^{-1}(x)$ passes through the point $(4\pi - 1, 2\pi)$, we know that $g^{-1}(4\pi - 1) = 2\pi$.

It follows that $(g^{-1})'(4\pi - 1) = \dfrac{1}{g'(2\pi)}$.

The expression for $g'(x)$ from part (i) implies that $g'(2\pi) = 2 + \sin(2\pi) = 2$. Therefore,

$$(g^{-1})'(4\pi - 1) = \frac{1}{g'(2\pi)} = \frac{1}{2}.$$

Since the tangent line passes through the point $(4\pi - 1, 2\pi)$ its equation is

$$\boxed{y - 2\pi = \frac{1}{2}(x - (4\pi - 1))}$$

2. (25 pts) Parts (a) and (b) are unrelated.

   (a) If a substance undergoing exponential decay has a half-life of 50 years, how many years would it take for a sample of that substance to decay to 1 percent of its original amount?

Solution:

Since the substance is undergoing exponential decay, the amount of the substance at time t years can be represented by $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$ is the amount of the substance at time $t = 0$ and k is the relative rate of change.

Since the half-life of the substance is 50 years, $y(50) = \frac{1}{2}\,y_0$, which implies the following:

$$y(50) = \frac{1}{2}\,y_0 = y_0 e^{50k}$$

$$\frac{1}{2} = e^{50k}$$

$$50k = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$k = -\frac{\ln 2}{50}$$

Therefore, the exponential decay model becomes $y(t) = y_0 e^{-(\ln 2/50)t}$.

The goal is to determine the time t at which $y(t) = 0.01 y_0$.

$$y(t) = 0.01 y_0 = y_0 e^{-(\ln 2/50)t}$$

$$0.01 = e^{-(\ln 2/50)t}$$

$$-(\ln 2/50)t = \ln(0.01) = -\ln(100)$$

$$t = \boxed{\frac{50\ln(100)}{\ln 2}\text{ years}}$$

(b) Consider the function $p(t) = p_0 e^{kt}$, which represents an exponential growth model for a population, where the constant p_0 represents the initial population size and the constant k represents the population's relative growth rate. Suppose $p(10) = 2$ and $p(50) = 6$.

i. Find the value of k.
ii. Find the value of p_0.

Solution:

The two given data points lead to the following system of two equations and two unknowns:

$$\begin{aligned} (t,p) = (10,2): &\qquad 2 = p_0 e^{10k} \\ (t,p) = (50,6): &\qquad 6 = p_0 e^{50k} \end{aligned}$$

Divide the bottom equation by the top equation to obtain the following:

$$\begin{aligned} \frac{p_0 e^{50k}}{p_0 e^{10k}} &= \frac{6}{2} \\ e^{40k} &= 3 \\ 40k &= \ln 3 \\ k &= \frac{\ln 3}{40} \end{aligned}$$

Therefore, the exponential growth model becomes $p(t) = p_0 e^{(\ln 3/40)t}$. Either of the two given data points can be used in this equation to determine the value of p_0. Using the point $(10, 2)$ produces the following:

$$\begin{aligned} 2 = p_0 e^{(\ln 3/40)(10)} &= p_0 e^{\ln 3/4} \\ p_0 &= \boxed{2e^{-\ln 3/4}} \end{aligned}$$

Alternatively, use the point $(50, 6)$:

$$\begin{aligned} 6 = p_0 e^{(\ln 3/40)(50)} &= p_0 e^{5\ln 3/4} \\ p_0 &= \boxed{6e^{-5\ln 3/4}} \end{aligned}$$

The two results are equivalent.

3. (26 pts) Evaluate the following derivatives using properties of logarithms and/or logarithmic differentiation. Do **not** fully simplify your answers, although they must be expressed as functions of x.

(a) $\dfrac{d}{dx}\left[\ln\left(\dfrac{(10-\cos^2 x)\sqrt{x^4+6}}{e^{x\sin x}}\right)\right]$

Solution:

$$\begin{aligned}
\frac{d}{dx}\left[\ln\left(\frac{(10-\cos^2 x)\sqrt{x^4+6}}{e^{x\sin x}}\right)\right] &= \frac{d}{dx}\left[\ln\left[(10-\cos^2 x)\right]+\ln\left[(x^4+6)^{1/2}\right]-\ln\left[e^{x\sin x}\right]\right] \\
&= \frac{d}{dx}\left[\ln\left[(10-\cos^2 x)\right]+\frac{1}{2}\ln(x^4+6)-x\sin x\right] \\
&= \frac{(-2\cos x)(-\sin x)}{10-\cos^2 x}+\frac{1}{2}\cdot\frac{4x^3}{x^4+6}-(x\cos x+\sin x) \\
&= \boxed{\frac{2\cos x\sin x}{10-\cos^2 x}+\frac{2x^3}{x^4+6}-x\cos x-\sin x}
\end{aligned}$$

(b) $\dfrac{d}{dx}\left[\left(e^x+e^{-x}\right)^x\right]$

Solution:

Let $y=(e^x+e^{-x})^x$.

$$\begin{aligned}\ln y &= \ln\left[\left(e^x+e^{-x}\right)^x\right]\\ &= x\ln\left(e^x+e^{-x}\right)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[\ln y] &= \frac{d}{dx}\left[x\ln\left(e^x+e^{-x}\right)\right]\\ \frac{1}{y}\cdot\frac{dy}{dx} &= x\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}+\ln\left(e^x+e^{-x}\right)\\ \frac{dy}{dx} &= y\left[x\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}+\ln\left(e^x+e^{-x}\right)\right]\\ \frac{dy}{dx} &= \boxed{\left(e^x+e^{-x}\right)^x\cdot\left[x\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}+\ln\left(e^x+e^{-x}\right)\right]}\end{aligned}$$

4. (26 pts) Evaluate the following integrals.

(a) $\displaystyle\int_1^2 \frac{2^x}{9-2^x}\,dx$

Solution:

Let $u = 9 - 2^x$, which implies that $du = -2^x \ln 2\,dx$.

$x = 1 \quad \Rightarrow \quad u = 9 - 2^1 = 7$

$x = 2 \quad \Rightarrow \quad u = 9 - 2^2 = 5$

$$\int_1^2 \frac{2^x}{9-2^x}\,dx = -\frac{1}{\ln 2}\int_7^5 \frac{du}{u} = \frac{1}{\ln 2}\int_5^7 \frac{du}{u} = \boxed{\frac{\ln 7 - \ln 5}{\ln 2}}$$

(b) $\displaystyle\int \frac{x}{x-1}\,dx$

Solution:

Let $u = x - 1$, which implies that $du = dx$ and $x = u + 1$.

$$\int \frac{x}{x-1}\,dx = \int \frac{u+1}{u}\,du = \int du + \int \frac{du}{u} = u + \ln|u| + C = \boxed{x - 1 + \ln|x-1| + C}$$

END OF EXAM

Your Initials _________

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.