APPM 1345
Exam 2
Spring 2024

Name		
Instructor	Richard McNamara	**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

1. (25 pts) Parts (a) and (b) are unrelated.

   (a) Find the average value f_{ave} of the function $f(x) = 9 - x^2$ on the interval $[0, 3]$, and find all values of c on $[0, 3]$ for which $f(c) = f_{\text{ave}}$.

   Solution:

$$\begin{aligned} f_{ave} &= \frac{1}{3}\int_0^3 (9 - x^2)\,dx = \frac{1}{3}\left[9x - \frac{x^3}{3}\right]\Bigg|_0^3 \\ &= \frac{1}{3}(27 - 9) = \boxed{6} \end{aligned}$$

$$\begin{aligned} f(c) &= f_{\text{ave}} \\ 9 - c^2 &= 6 \\ c^2 &= 3 \\ c &= \pm\sqrt{3} \end{aligned}$$

   Since $-\sqrt{3}$ is not on the interval $[0, 3]$, the only solution is $c = \boxed{\sqrt{3}}$

   (b) Evaluate the following derivatives.

   i. $\displaystyle \frac{d}{dx}\int_x^1 \sqrt{1+t^2}\,dt$

   Solution:

$$\frac{d}{dx}\int_x^1 \sqrt{1+t^2}\,dt = -\frac{d}{dx}\int_1^x \sqrt{1+t^2}\,dt = \boxed{-\sqrt{1+x^2}}$$

   ii. $\displaystyle \frac{d}{dx}\int_{\cos x}^{\sin x} \frac{1}{t+2}\,dt$

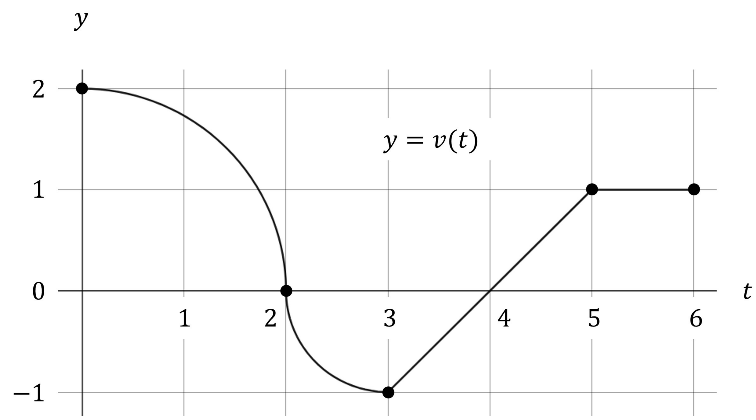
   Solution:

$$\begin{aligned} \frac{d}{dx}\int_{\cos x}^{\sin x} \frac{1}{t+2}\,dt &= \frac{d}{dx}\int_a^{\sin x} \frac{1}{t+2}\,dt - \frac{d}{dx}\int_a^{\cos x} \frac{1}{t+2}\,dt \\ &= \frac{1}{\sin x + 2}\cdot\cos x - \frac{1}{\cos x + 2}\cdot(-\sin x) \\ &= \boxed{\frac{\cos x}{\sin x + 2} + \frac{\sin x}{\cos x + 2}} \end{aligned}$$

2. (20 pts) The graph below represents a particle's velocity function $v(t)$, and the corresponding position function for $0 \leq t \leq 6$ is

$$s(t) = \int_0^t v(u)\, du$$

The graph of $v(t)$ consists of two quarter-circles and two line segments.



(a) Determine the particle's position at time $t = 0$ and at time $t = 3$.

(b) What is total distance traveled by the particle between $t = 0$ and $t = 3$?

(c) When is the particle moving in the positive direction? Express your answer using interval notation.

(d) When is the particle's acceleration positive? Express your answer using interval notation.

Solution:

(a) $s(0) = \displaystyle\int_0^0 v(u)\, du = \boxed{0}$

Let A_1 represent the area of the quarter circle of radius 2 centered at $(0, 0)$, and let Let A_2 represent the area of the quarter circle of radius 1 centered at the point $(3, 0)$.

$$s(3) = \int_0^3 v(u)\, du = A_1 - A_2 = \frac{1}{4}\pi(2)^2 - \frac{1}{4}\pi(1)^2 = \pi - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$$

(b) $D = A_1 + A_2 = \pi + \dfrac{\pi}{4} = \boxed{\dfrac{5\pi}{4}}$

(c) The particle is moving in the positive direction when its velocity is positive: $\boxed{[0, 2) \cup (4, 6]}$

(d) The particle's acceleration is positive when its velocity is increasing: $\boxed{(3, 5)}$

3. (27 pts) Parts (a) and (b) are unrelated.

(a) Evaluate the following integrals. Fully simplify your answers.

i. $\displaystyle\int \frac{x}{\sqrt{3x^2+1}}\,dx$

Solution: Use u-substitution with $u = 3x^2 + 1$, so that $du = 6x\,dx$.

$$\int \frac{x}{\sqrt{3x^2+1}}\,dx = \int \left(3x^2+1\right)^{-1/2}(x\,dx) = \int u^{-1/2}\left(\frac{1}{6}\,du\right)$$

$$= \frac{1}{6}\cdot\left(2u^{1/2}\right) + C = \boxed{\frac{1}{3}\sqrt{3x^2+1} + C}$$

ii. $\displaystyle\int \frac{\sin\left(3x^{1/3}\right)}{x^{2/3}}\,dx$

Solution: Use u-substitution with $u = 3x^{1/3}$, so that $du = x^{-2/3}\,dx$.

$$\int \frac{\sin\left(3x^{1/3}\right)}{x^{2/3}}\,dx = \int \sin u\,du = -\cos u + C = \boxed{-\cos\left(3x^{1/3}\right) + C}$$

(b) Suppose $g(x)$ is a continuous function such that $\displaystyle\int_1^7 g(x)\,dx = 15$. Find the value of $\displaystyle\int_0^2 g(3x+1)\,dx$. (*Hint:* Apply u-substitution.)

Solution:

Use u-substitution with $u = 3x + 1$, so that $du = 3\,dx$.

$x = 0$ implies that $u = 1$ and $x = 2$ implies that $u = 7$.

$$\int_0^2 g(3x+1)\,dx = \int_1^7 g(u)\cdot\left(\frac{1}{3}\,du\right)$$

$$= \frac{1}{3}\int_1^7 g(u)\,du$$

$$= \frac{1}{3}\cdot 15 = \boxed{5}$$

4. (28 pts) Parts (a) and (b) are unrelated.

(a) Consider the function $h(x) = \cos^2 x$ on the interval $I = [0, \pi/2]$.

i. Determine the numerical value of the Riemann sum L_2 for $h(x)$ on I using **left** endpoints and 2 equally-sized subintervals. Fully simplify your answer.

ii. Write an expression for the general Riemann sum L_n for $h(x)$ on I using **left** endpoints and n equally-sized subintervals. Express your answer using sigma notation.

Solution:

i. $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi/2 - 0}{2} = \dfrac{\pi}{4}$

Since left endpoints are being used and the subintervals are of equal size, we have $x_0 = 0$ and $x_1 = \pi/4$.

$$\begin{aligned}
L_2 &= [h(x_0) + h(x_1)]\,\Delta x \\
&= \left[\cos^2(0) + \cos^2(\pi/4)\right] \cdot \frac{\pi}{4} \\
&= \left[1^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right] \cdot \frac{\pi}{4} \\
&= \frac{3}{2} \cdot \frac{\pi}{4} = \boxed{\frac{3\pi}{8}}
\end{aligned}$$

ii. $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi/2 - 0}{n} = \dfrac{\pi}{2n}$

Since left endpoints are being used and the subintervals are of equal size, we have

$$x_{i-1} = a + (i-1)\Delta x = \frac{(i-1)\pi}{2n}$$

$$\begin{aligned}
L_n &= \sum_{i=1}^{n} h(x_{i-1})\,\Delta x \\
&= \sum_{i=1}^{n} \cos^2(x_{i-1}) \cdot \frac{\pi}{2n} \\
&= \boxed{\frac{\pi}{2n} \sum_{i=1}^{n} \cos^2\left[\frac{(i-1)\pi}{2n}\right]}
\end{aligned}$$

(b) Suppose the following expression is a Riemann sum for a continuous function $u(x)$ on the interval $[-1, 2]$:

$$R_n = \sum_{i=1}^{n} \left[\left(\frac{3i}{n} \right)^2 + 1 \right] \left(\frac{3}{n} \right)$$

Find the numerical value of $\int_{-1}^{2} u(x)\,dx$ by evaluating the appropriate limit of R_n. Do not use a Dominance of Powers argument when evaluating the limit. Fully simplify your answer.

Solution:

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \left[\left(\frac{3i}{n} \right)^2 + 1 \right] \left(\frac{3}{n} \right) \\ &= \frac{3}{n} \sum_{i=1}^{n} \left[\frac{9}{n^2} \cdot i^2 + 1 \right] \\ &= \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1 \end{aligned}$$

Applying the appropriate formula on the cover page of the exam produces the following:

$$\begin{aligned} R_n &= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot n \\ &= \frac{9}{2} \cdot \frac{(n+1)(2n+1)}{n^2} + 3 \\ &= \frac{9}{2} \cdot \frac{2n^2 + 3n + 1}{n^2} + 3 \\ &= \frac{9}{2} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) + 3 \end{aligned}$$

$$\begin{aligned} \int_{-1}^{2} u(x)\,dx &= \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[\frac{9}{2} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) + 3 \right] \\ &= \frac{9}{2}\,(2 + 0 + 0) + 3 = 9 + 3 = \boxed{12} \end{aligned}$$

END OF EXAM