APPM 1345
Exam 1
Spring 2024

Name		
Instructor	Richard McNamara	**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

1. (29 pts) Parts (a) and (b) are unrelated.

   (a) Find the most general form of $u(x)$ such that $u''(x) = \sin x + x^{3/4} + 1$.

   Solution:

   $$u'(x) = -\cos x + \frac{x^{7/4}}{7/4} + x + C_1 = -\cos x + \frac{4}{7}x^{7/4} + x + C_1$$

   $$u(x) = -\sin x + \frac{4}{7} \cdot \frac{x^{11/4}}{11/4} + \frac{x^2}{2} + C_1 x + C_2 = -\sin x + \frac{4}{7} \cdot \frac{4}{11} x^{11/4} + \frac{x^2}{2} + C_1 x + C_2$$

   $$u(x) = \boxed{-\sin x + \frac{16}{77}x^{11/4} + \frac{x^2}{2} + C_1 x + C_2}$$

   (b) Consider a particle that is moving along a linear path. Let t represent the time in seconds, let $s(t)$ represent the position in meters, let $v(t)$ represent the velocity in $\mathrm{m/s}$, and let $a(t)$ represent the acceleration in $\mathrm{m/s^2}$. Suppose the particle is accelerating at a constant rate of $a = 4\ \mathrm{m/s^2}$, its initial velocity is $v(0) = 2\ \mathrm{m/s}$, and its initial position is $s(0) = 6$ meters. How many seconds would it take for the particle to move from a position of $s = 6$ meters to a position of $s = 30$ meters? Show the full derivation of your results (do not simply use a formula from a different course).

   Solution:

   $a(t) = 4$

   The velocity function is an antiderivative of the acceleration function.

   $v(t) = 4t + C_1$ and $v(0) = 2$, so $C_1 = 2$. Therefore, $v(t) = 4t + 2$.

   The position function is an antiderivative of the velocity function.

   $s(t) = 2t^2 + 2t + C_2$ and $s(0) = 6$, so $C_2 = 6$. Therefore, $s(t) = 2t^2 + 2t + 6$.

   We need to solve for the value of t for which $s(t) = 30$.

   $s(t) = 2t^2 + 2t + 6 = 30$

   $2t^2 + 2t - 24 = 0$

   $2(t^2 + t - 12) = 2(t-3)(t+4) = 0 \quad \Rightarrow \quad t = -4, 3$

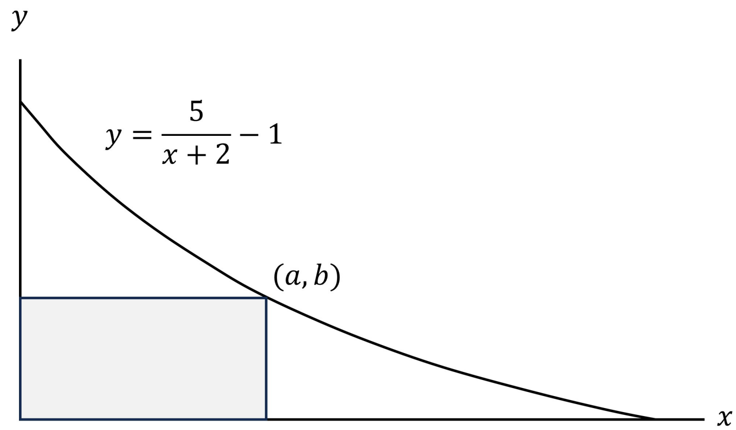
   Movement in the positive direction with an acceleration that is always positive implies that t must be positive.

   $\boxed{t = 3 \text{ seconds}}$

2. (22 pts) The rectangle shown has one side on the positive x-axis, one side on the positive y-axis, and its upper right corner at the point (a, b), which lies on the curve

$$y = \frac{5}{x+2} - 1$$

Identify the point (a, b) that produces the rectangle with the largest area, and use the Second Derivative Test to confirm that your result is a local maximum value of the area function.



Solution:

The area of the rectangle whose upper right corner is at the point (x, y), where $y = \dfrac{5}{x+2} - 1$ is

$$A = xy = x\left(\frac{5}{x+2} - 1\right) = \frac{5x}{x+2} - x \quad \Rightarrow \quad A(x) = \frac{5x}{x+2} - x$$

Next, we find the critical numbers of the area function $A(x)$ by setting $A'(x) = 0$.

$$A'(x) = \frac{d}{dx}\left[\frac{5x}{x+2} - x\right] = \frac{(x+2)(5) - (5x)(1)}{(x+2)^2} - 1 = \frac{10}{(x+2)^2} - 1 = 0$$

$$(x+2)^2 = 10 \quad \Rightarrow \quad x + 2 = \sqrt{10} \quad \Rightarrow \quad x = \sqrt{10} - 2$$

$$y = \frac{5}{x+2} - 1 = \frac{5}{(\sqrt{10} - 2) + 2} - 1 = \frac{5}{\sqrt{10}} - 1$$

Therefore, the maximum area corresponds to $\boxed{(a, b) = \left(\sqrt{10} - 2, \dfrac{5}{\sqrt{10}} - 1\right)}$

$$A''(x) = \frac{d}{dx}\left[\frac{10}{(x+2)^2} - 1\right] = -\frac{20}{(x+2)^3} < 0 \quad (\textit{note: } x > 0)$$

Therefore, since $A'(\sqrt{10} - 2) = 0$ and $A''(\sqrt{10} - 2) < 0$ the Second Derivative Test indicates that the result is a local *maximum* value of the area function $A(x)$.

3. (23 pts) Suppose Newton's Method is used to estimate the value of a root of $y = p(x) = 0.05(2x^3 - 3x^2 - 12x + 5)$.

(a) Write the expression for Newton's Method for the specified function $p(x)$. Your answer should be an expression for x_{n+1} in terms of x_n.

Solution:

The general equation for Newton's Method when applied to a function $p(x)$ is $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)}$.

$p'(x) = 0.05(6x^2 - 6x - 12) = (0.05)(6)(x^2 - x - 2)$

Therefore, for the given function $p(x)$, the expression for Newton's Method is

$$\boxed{x_{n+1} = x_n - \frac{2{x_n}^3 - 3{x_n}^2 - 12x_n + 5}{6({x_n}^2 - x_n - 2)}}$$

(b) Use the result from part (a) with a value of $x_0 = 1$ to determine the corresponding value of x_1.

Solution:

Set $n = 0$ in the resulting equation from part (a) to produce the following equation for x_1.

$$x_1 = x_0 - \frac{2{x_0}^3 - 3{x_0}^2 - 12x_0 + 5}{6({x_0}^2 - x_0 - 2)}$$

Substituting $x_0 = 1$ into the preceding equation leads to the following result.

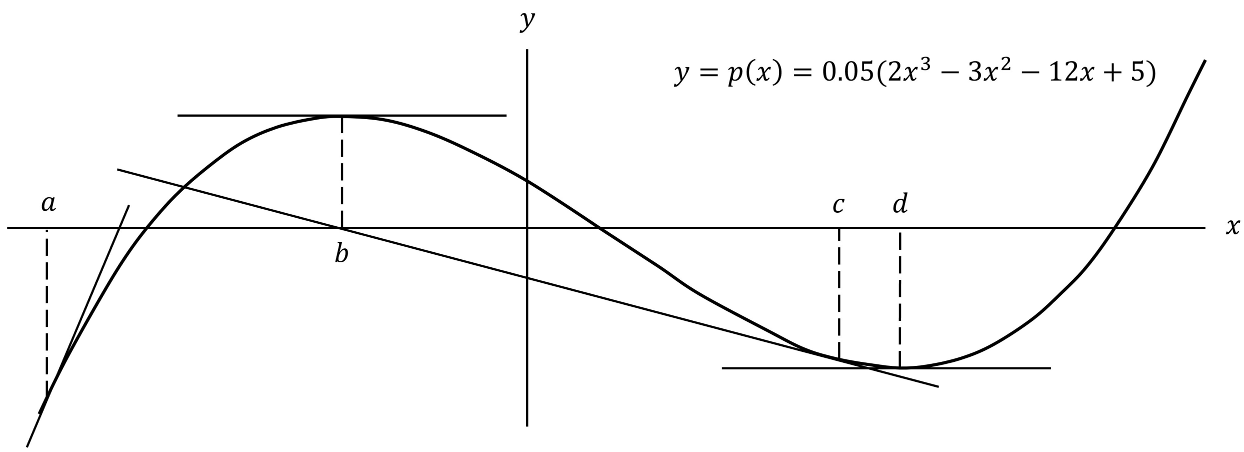
$$x_1 = 1 - \frac{2 \cdot 1^3 - 3 \cdot 1^2 - 12 \cdot 1 + 5}{6(1^2 - 1 - 2)} = 1 - \frac{2 - 3 - 12 + 5}{6(1 - 1 - 2)} = 1 - \frac{-8}{(6)(-2)}$$

$$x_1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

(c) The following graph of $y = p(x)$ labels four particular x values as constants a through d. The line tangent to $y = p(x)$ at each of those four x values is also shown.

i. Determine the actual numerical values of the constants b and d, which correspond to horizontal tangent lines.

ii. Consider using each of the four labeled x values (a through d) as the value of x_0 in Newton's Method. For which of those choices would Newton's Method fail to converge? List all that apply, and in each case, briefly explain why it would not converge.



Solution:

i. Part (a) indicates that $p'(x) = (0.05)(6)(x^2 - x - 2) = 0.3(x^2 - x - 2)$. The curve $y = p(x)$ has a horizontal tangent at $x = b$ and at $x = d$, the numerical values of which can be found by setting $p'(x) = 0$ and solving for x.

$p'(x) = 0.3(x^2 - x - 2) = 0.3(x - 2)(x + 1) = 0$

The solutions are $x = -1$ and $x = 2$, so that $\boxed{b = -1}$ and $\boxed{d = 2}$

ii. Newton's Method would not converge for $\boxed{x_0 = b}$ and $\boxed{x_0 = d}$ because the curve $y = p(x)$ has horizontal tangent lines at those locations.

Newton's Method would not converge for $\boxed{x_0 = c}$ because its tangent line leads to a value of $x_1 = b$, and there is a horizontal tangent line at that location.

4. (26 pts) Let $f(x) = (x-1)\sin x + \cos x$ on the interval $[0, \pi]$. Answer the following for the specified interval.

(a) Identify all critical numbers of $f(x)$.

(b) For which values of x is $f(x)$ increasing and for which values of x is $f(x)$ decreasing? Express your answers using interval notation.

(c) Identify the x-coordinate of each local maximum and minimum value of $f(x)$ (if any). Use the First Derivative Test to classify each one.

Solution:

(a) $f'(x) = [(x-1)\cos x + \sin x] - \sin x = (x-1)\cos x$

Since $f'(x)$ exists for all real values of x, the only critical numbers of $f(x)$ are values of x such that $f'(x) = 0$.

$f'(x) = (x-1)\cos x = 0$, which implies that the critical numbers of $f(x)$ are $\boxed{x = 1, \pi/2}$

(b) The term $(x-1)$ is negative for x on the interval $[0, 1)$ and is positive for x on the interval $(1, \pi]$.

The term $\cos x$ is positive for x on the interval $[0, \pi/2)$ and is negative for x on the interval $(\pi/2, \pi]$.

A pictorial representation of the preceding information is provided in the following table, along with a summary of the sign of $f'(x)$ on various subintervals of $[0, \pi]$.

$x-1$	$-$	$+$	$+$
$\cos x$	$+$	$+$	$-$
$f'(x) = (x-1)\cos x$	$-$	$+$	$-$

x

$x = 0$ $\quad x = 1$ $\quad x = \frac{\pi}{2}$ $\quad x = \pi$

The bottom row of the table indicates that $f(x)$ is increasing on $\boxed{(1, \pi/2)}$ and is decreasing on $\boxed{[0, 1) \cup (\pi/2, \pi]}$

(c) The First Derivative Test indicates that because $f(x)$ is continuous and the sign of $f'(x)$ transitions from negative to positive at $x = 1$, the function $f(x)$ has a $\boxed{\text{local minimum at } x = 1}$ and because the sign of $f'(x)$ transitions from positive to negative at $x = \pi/2$, the function $f(x)$ has a $\boxed{\text{local maximum at } x = \pi/2}$

END OF EXAM

ADDITIONAL BLANK SPACE
If you write a solution here, please clearly indicate the problem number.