1. (44 pts) Evaluate the following expressions. Fully simplify your answers.

(a) $\lim_{x \to 0} \dfrac{\sin(x/2)}{e^{2x} - 1}$

**Solution:**

$$\underbrace{\lim_{x \to 0} \frac{\sin(x/2)}{e^{2x} - 1}}_{0/0} \overset{LH}{=} \lim_{x \to 0} \frac{\frac{1}{2}\cos(x/2)}{2e^{2x}} = \frac{\frac{1}{2}\cos 0}{2e^0} = \frac{\frac{1}{2}}{2} = \boxed{\frac{1}{4}}$$

(b) $\dfrac{d}{dx} \displaystyle\int_0^{x^2} e^t \cos t \, dt$

**Solution:** By FTC-1 and the chain rule:

$$\frac{d}{dx} \int_0^{x^2} e^t \cos t \, dt = \boxed{2xe^{x^2} \cos\left(x^2\right)}$$

(c) $\dfrac{d}{dx} \left((\sec x)^x\right)$

**Solution:**

$$\begin{aligned} y &= (\sec x)^x \\ \ln y &= \ln\left((\sec x)^x\right) \\ &= x \ln\left(\sec x\right) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= x \cdot \frac{\sec x \tan x}{\sec x} + \ln\left(\sec x\right) \\ &= x \tan x + \ln\left(\sec x\right) \\ \frac{dy}{dx} &= \boxed{(\sec x)^x \left(x \tan x + \ln\left(\sec x\right)\right)} \end{aligned}$$

(d) $\displaystyle\int \frac{\tan\theta}{\cos^2\theta} \, d\theta$

**Solution:** Let $u = \tan\theta$, $du = \sec^2\theta \, d\theta$.

$$\begin{aligned} \int \frac{\tan\theta}{\cos^2\theta} \, d\theta &= \int \tan\theta \sec^2\theta \, d\theta \\ &= \int u \, du = \frac{u^2}{2} + C = \boxed{\frac{1}{2}\tan^2\theta + C} \end{aligned}$$

**Alternate Solution:** Let $u = \sec\theta$, $du = \sec\theta\tan\theta\,d\theta$.

$$\int \frac{\tan\theta}{\cos^2\theta}\,d\theta = \int \tan\theta\sec^2\theta\,d\theta$$
$$= \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{1}{2}\sec^2\theta + C}$$

**Alternate Solution:**

$$\int \frac{\tan\theta}{\cos^2\theta}\,d\theta = \int \frac{\frac{\sin\theta}{\cos\theta}}{\cos^2\theta}\,d\theta$$
$$= \int \frac{\sin\theta}{\cos^3\theta}\,d\theta$$

Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$.

$$\int \frac{\tan\theta}{\cos^2\theta}\,d\theta = \int -\frac{du}{u^3} = \int -u^{-3}\,du$$
$$= \frac{1}{2}u^{-2} + C = \boxed{\frac{1}{2}(\cos\theta)^{-2} + C} = \boxed{\frac{1}{2}\sec^2\theta + C}$$

(e) $\displaystyle\sum_{i=1}^{6} \ln\left(\frac{i+3}{i+2}\right)$ (Write your answer as a single log expression.)

**Solution:**

$$\sum_{i=1}^{6} \ln\left(\frac{i+3}{i+2}\right) = \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{6}{5}\right) + \cdots + \ln\left(\frac{9}{8}\right)$$
$$= \ln\left(\frac{\not{4}}{3}\cdot\frac{\not{5}}{\not{4}}\cdot\frac{\not{6}}{\not{5}}\cdots\frac{9}{\not{8}}\right) = \ln\left(\frac{9}{3}\right) = \boxed{\ln 3}$$

2. (12 pts) A particle is moving along a straight line. The position function of the particle in meters after $t$ seconds is given by

$$s(t) = \frac{t^3}{3} - \frac{5t^2}{2} + 6t, \quad 1 \le t \le 5.$$

(a) Find the particle's instantaneous velocity $v(t)$ at $t = 4$ seconds.

(b) What is the average value of the acceleration $a(t)$ on the interval $[1, 5]$?

**Solution:**

(a)

$$s(t) = \frac{t^3}{3} - \frac{5t^2}{2} + 6t$$
$$v(t) = s'(t) = t^2 - 5t + 6$$
$$v(4) = 16 - 20 + 6 = \boxed{2 \text{ m/sec}}$$

(b) Note that $a(t) = v'(t)$, so $\int a(t)\,dt = v(t) + C$.

$$\begin{aligned} f_{ave} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ a_{ave} &= \frac{1}{5-1}\int_1^5 a(t)\,dt \\ &= \frac{1}{4}\left[v(t)\right]_1^5 \\ &= \frac{1}{4}\left[t^2 - 5t + 6\right]_1^5 \\ &= \frac{1}{4}\left[\left(5^2 - 5^2 + 6\right) - (1 - 5 + 6)\right] \\ &= \frac{1}{4}(6-2) = \boxed{1 \text{ m/sec}^2}. \end{aligned}$$

3. (28 pts) Let $g(x) = \dfrac{x}{1+4x}$.

(a) What is the domain of the function? Express your answer in interval notation.

**Solution:** The function is not defined when the denominator equals zero at $x = -\frac{1}{4}$, so the domain is $\boxed{\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)}$.

(b) Find $g'(x)$ and simplify.

**Solution:** By the quotient rule:

$$g'(x) = \frac{(1+4x) - 4x}{(1+4x)^2} = \boxed{\frac{1}{(1+4x)^2}}.$$

(c) Find the inverse function $g^{-1}(x)$.

**Solution:** First solve for $x$.

$$\begin{aligned} y &= \frac{x}{1+4x} \\ y + 4xy &= x \\ 4xy - x &= -y \\ x(4y-1) &= -y \\ x &= \frac{-y}{4y-1} \end{aligned}$$

Then swap $x$ and $y$.

$$g^{-1}(x) = \boxed{\frac{-x}{4x-1}} = \boxed{\frac{x}{1-4x}}$$

(d) Evaluate $\displaystyle\int_1^6 \frac{1}{1+4x}\,dx$.

**Solution:** Let $u = 1 + 4x$, $du = 4\,dx$.

$$\int_1^6 \frac{dx}{1+4x} = \int_5^{25} \frac{1}{4}\frac{du}{u} = \frac{1}{4}\ln|u|\Big]_5^{25} = \frac{1}{4}\left(\ln 25 - \ln 5\right) = \frac{1}{4}\ln\frac{25}{5} = \boxed{\frac{1}{4}\ln 5}$$

4. (12 pts) The Shiveluch volcano on the Kamchatka peninsula is currently erupting and forming a lava dome in the shape of a hemisphere.

   (a) When the radius of the dome is 10 meters, it is increasing at a rate of 2 meters/hour. How fast is the volume of the dome changing? (The volume of a hemisphere is $V = \frac{2}{3}\pi r^3$.)

   (b) Assume that the volume's rate of change remains constant. Find the radius when it is increasing at a rate of 1 meter/hour.

**Solution:**

(a) It is given that $dr/dt = 2$ m/hr when $r = 10$ m. We wish to find $dV/dt$.

$$\begin{aligned} V &= \frac{2}{3}\pi r^3 \\ \frac{dV}{dt} &= 2\pi r^2 \frac{dr}{dt} \\ &= 2\pi 10^2 \cdot 2 = \boxed{400\pi \text{ m}^3\text{/hr}} \end{aligned}$$

(b) Now find $r$ when $dr/dt = 1$ m/hr.

$$\begin{aligned} \frac{dV}{dt} &= 2\pi r^2 \frac{dr}{dt} \\ 400\pi &= 2\pi r^2 \cdot 1 \\ 200 &= r^2 \\ r &= \sqrt{200} = \boxed{10\sqrt{2} \text{ m}} \end{aligned}$$

5. (28 pts) The following two problems are not related.

   (a) Let $f(x) = \dfrac{\sin^{-1}(x)}{x}$.

      i. Find the values of $f(-1)$ and $f(1)$.

      ii. Does $f(x)$ have any vertical asymptotes? If so, find them. Justify your answer using limits.

   **Solution:**

   i. $f(-1) = \dfrac{\sin^{-1}(-1)}{-1} = \dfrac{-\pi/2}{-1} = \boxed{\dfrac{\pi}{2}}$, $f(1) = \dfrac{\sin^{-1}(1)}{1} = \dfrac{\pi/2}{1} = \boxed{\dfrac{\pi}{2}}$

ii. The only potential vertical asymptote is at $x = 0$.

$$\lim_{x\to 0} \underbrace{\frac{\sin^{-1}(x)}{x}}_{0/0} \overset{LH}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1.$$

Because the function does not approach positive or negative infinity at $x = 0$, it has $\boxed{\text{no vertical asymptotes}}$.

(b) Let $h(x) = \sinh(\ln x)$.

i. Find $h'(x)$.

ii. Find an equation of the line tangent to the curve $y = h(x)$ at $x = 3$. Write your fully simplified answer in slope-intercept form with no hyperbolic functions.

**Solution:**

i. $h'(x) = \boxed{\frac{1}{x}\cosh(\ln x)}$

ii.

$$h(3) = \sinh(\ln 3) = \frac{e^{\ln 3} - e^{-\ln 3}}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{\frac{8}{3}}{2} = \frac{4}{3}$$

$$h'(3) = \frac{1}{3}\cosh(\ln 3) = \frac{1}{3}\cdot\frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{1}{3}\cdot\frac{3+\frac{1}{3}}{2} = \frac{1}{3}\cdot\frac{\frac{10}{3}}{2} = \frac{5}{9}$$

The tangent line is

$$\begin{aligned} y &= h(3) + h'(3)(x-3) \\ &= \frac{4}{3} + \frac{5}{9}(x-3) \\ &\boxed{y = \frac{5}{9}x - \frac{1}{3}}. \end{aligned}$$

6. (12 pts) A sample of the radioactive element Unobtainium decayed by $10\%$ in one day. In hours, how long did it take for the sample to decay by $3\%$?

**Solution:**

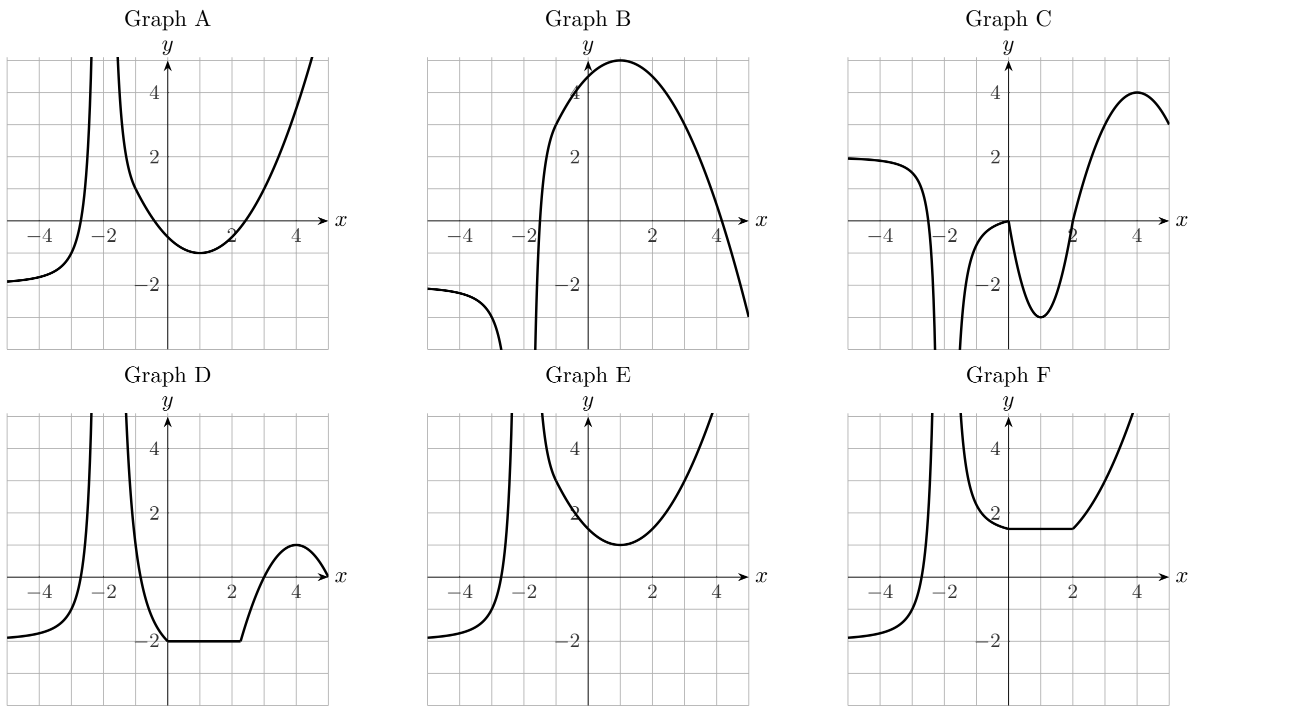
Let $m(t) = m_0 e^{kt}$ represent the mass of the radioactive substance. It is given that after 24 hours, $m(24) = 0.9m_0$. First find $k$.

$$\begin{aligned} m(24) = 0.9m_0 &= m_0 e^{24k} \\ 0.9 &= e^{24k} \\ \ln 0.9 &= \ln\left(e^{24k}\right) = 24k \\ k &= \frac{\ln 0.9}{24} \end{aligned}$$

Now find $t$ when $m(t) = 0.97m_0$.

$$m(t) = 0.97m_0 = m_0 e^{kt}$$
$$0.97 = e^{kt}$$
$$\ln 0.97 = \ln\left(e^{kt}\right) = kt$$
$$t = \frac{\ln 0.97}{k} = \boxed{\frac{24 \ln 0.97}{\ln 0.9} \text{ hours}}$$

7. (14 pts) Consider the six graphs A, B, C, D, E, and F shown below. No justification is necessary for the following questions.



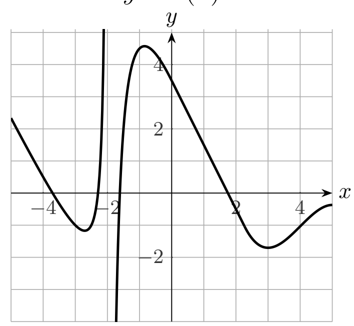

(a) Which of the six graphs satisfies all of the following conditions?

- $\lim_{x \to -\infty} f(x) = -2$
- $\lim_{x \to -2} f(x) = \infty$
- $f'(1) = 0$ and $f''(1) > 0$
- the line $y = 2x - 3$ is tangent to $f$ at $x = 3$

**Solution:** Graph E

(b) Which of the six graphs is the derivative graph of the function $y = r(x)$ shown below?

$y = r(x)$




**Solution:** Graph D