APPM 1345
Exam 3
Spring 2023

Name		
Instructor	Richard McNamara	**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

Formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

1. (23 pts) Parts (a) and (b) are unrelated.

(a) Find the inverse function of $g(x) = 6x^5 - 1$.

Solution:

$y = g(x) = 6x^5 - 1$

$6x^5 = y + 1$

$x^5 = \dfrac{y+1}{6}$

$x = \sqrt[5]{\dfrac{y+1}{6}}$

Reverse the roles of x and y to get $\boxed{y = g^{-1}(x) = \sqrt[5]{\dfrac{x+1}{6}}}$

(b) Consider the function $f(x) = 2x^5 + x^3 + 3x + 2$.

i. Explain why f is invertible, based on its derivative.

ii. Find an equation of the line that is tangent to the curve $y = f^{-1}(x)$ at the point $(8, 1)$.

Solution:

i. $f'(x) = 10x^4 + 3x^2 + 3$, which is positive for $-\infty < x < \infty$.

Therefore, $f(x)$ is a monotone increasing function, which implies that it is invertible.

ii. The slope of the line that is tangent to the curve $y = f^{-1}(x)$ at the point $(8, 1)$ is $(f^{-1})'(8)$.

Since $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, we know that $(f^{-1})'(8) = \dfrac{1}{f'(f^{-1}(8))}$.

Since the curve $y = f^{-1}(x)$ passes through the point $(8, 1)$, we know that $f^{-1}(8) = 1$.

It follows that $(f^{-1})'(8) = \dfrac{1}{f'(1)}$.

The expression for $f'(x)$ from part (i) implies that $f'(1) = 10 \cdot 1^4 + 3 \cdot 1^2 + 3 = 16$. Therefore,

$(f^{-1})'(8) = \dfrac{1}{f'(1)} = \dfrac{1}{16}$.

Since the tangent line passes through the point $(8, 1)$ its equation is $\boxed{y - 1 = \dfrac{1}{16}(x - 8)}$

2. (27 pts) Parts (a), (b) and (c) are unrelated.

(a) Suppose $1/3$ of a radioactive substance remains after decaying exponentially for 10 years. Find the half-life of the substance, including the correct unit of measurement. Fully support your answer.

Solution:

Since the substance is undergoing exponential decay, the amount of the substance at time t years can be represented by $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$ is the amount of the substance at time $t = 0$ and k is the relative rate of change. Therefore, for $t = 10$, we have the following:

$$y(10) = y_0 e^{10k} = \frac{y_0}{3} \quad \Rightarrow \quad 10k = \ln\left(\frac{1}{3}\right) \quad \Rightarrow \quad k = \frac{1}{10}\ln\left(\frac{1}{3}\right) = \frac{1}{10}(-\ln 3) = -\frac{\ln 3}{10}$$

$$y(t) = y_0 e^{-(\ln 3/10)t}$$

The half-life of the substance is the value of t for which $y(t) = \dfrac{y_0}{2}$, which is the solution of the following equation:

$$y_0 e^{-(\ln 3/10)t} = \frac{y_0}{2} \quad \Rightarrow \quad e^{-(\ln 3/10)t} = \frac{1}{2} \quad \Rightarrow \quad -\left(\frac{\ln 3}{10}\right)t = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$t = \boxed{\frac{10\ln 2}{\ln 3}\text{ years}}$$

(b) Identify all critical numbers of the function $h(x) = x^2\, 3^x$, if any.

Solution:

$$h'(x) = x^2\, 3^x \ln 3 + 2x\, 3^x = x\, 3^x(x\ln 3 + 2)$$

The critical numbers for this function are the values of x such that $h'(x) = 0$:

$$h'(x) = x\, 3^x(x\ln 3 + 2) = 0 \quad \Rightarrow \quad x = \boxed{0, -\frac{2}{\ln 3}}$$

(c) Rewrite the expression $e^{(5\ln 2)t}$ so that it includes no logarithmic terms.

Solution:

$$e^{(5\ln 2)t} = e^{5t\ln 2} = e^{\ln\left(2^{5t}\right)} = \boxed{2^{5t}}$$

3. (24 pts) Evaluate the following derivatives using properties of logarithms and/or logarithmic differentiation. Do **not** fully simplify your answers, although they must be expressed as functions of x.

(a) $\dfrac{d}{dx}\left[\ln\left(\dfrac{(x-2)^{3/2}(\cos x+2)}{\sqrt{x^2+4}}\right)\right], \quad x>2$

Solution:

$$\frac{d}{dx}\left[\ln\left(\frac{(x-2)^{3/2}(\cos x+2)}{\sqrt{x^2+4}}\right)\right] = \frac{d}{dx}\left[\ln\left[(x-2)^{3/2}\right] + \ln(\cos x+2) - \ln\left[(x^2+4)^{1/2}\right]\right]$$

$$= \frac{d}{dx}\left[\frac{3}{2}\ln(x-2) + \ln(\cos x+2) - \frac{1}{2}\ln(x^2+4)\right]$$

$$= \frac{3}{2}\cdot\frac{1}{x-2} + \frac{(-\sin x)}{\cos x+2} - \frac{1}{2}\cdot\frac{2x}{x^2+4} = \boxed{\frac{3}{2(x-2)} - \frac{\sin x}{\cos x+2} - \frac{x}{x^2+4}}$$

(b) $\dfrac{d}{dx}\left[(x^6+1)^{\sin x}\right]$

Solution:

Let $y=(x^6+1)^{\sin x}$.

$$\ln y = \ln\left[(x^6+1)^{\sin x}\right] = \sin x\,\ln(x^6+1)$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[\sin x\,\ln(x^6+1)\right]$$

$$\frac{y'}{y} = \sin x\cdot\frac{6x^5}{x^6+1} + \cos x\,\ln(x^6+1)$$

$$y' = y\left[\frac{6x^5\sin x}{x^6+1} + \cos x\,\ln(x^6+1)\right]$$

$$\frac{d}{dx}\left[(x^6+1)^{\sin x}\right] = \boxed{(x^6+1)^{\sin x}\left[\frac{6x^5\sin x}{x^6+1} + \cos x\,\ln(x^6+1)\right]}$$

4. (26 pts) Evaluate the following integrals. Fully simplify your answers.

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}(1-2\sqrt{x})}$

Solution:

Let $u = 1 - 2\sqrt{x} = 1 - 2x^{1/2}$, which implies that $du = -x^{-1/2}\,dx = -\dfrac{dx}{\sqrt{x}}$.

$$x = 4 \quad \Rightarrow \quad u = 1 - 2\sqrt{4} = -3$$
$$x = 9 \quad \Rightarrow \quad u = 1 - 2\sqrt{9} = -5$$

$$\int_4^9 \frac{dx}{\sqrt{x}(1-2\sqrt{x})} = -\int_{-3}^{-5} \frac{du}{u} = \int_{-5}^{-3} \frac{du}{u} = \ln|u| \Big|_{-5}^{-3} = \ln|-3| - \ln|-5| = \ln 3 - \ln 5 = \boxed{\ln\left(\frac{3}{5}\right)}$$

(b) $\displaystyle\int \cot x\,dx$

Solution:

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$$

Let $u = \sin x$, which implies that $du = \cos x\,dx$.

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sin x| + C}$$

END OF TEST

Your Initials _________

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.