APPM 1345
Exam 2
Spring 2023

Name		
Instructor Richard McNamara		**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

1. (21 pts) The following are related to the definite integal $A = \int_0^2 x^2 dx$.

(a) Approximate the value of A using a Riemann sum with a regular partition of $n = 4$ subintervals and right endpoints (that is, determine the value of R_4). Fully simplify your answer.

Solution:

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$R_4 = \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{3}{2}\right)^2 + 2^2\right] = \frac{1}{2}\left[\frac{1}{4} + 1 + \frac{9}{4} + 4\right] = \frac{1}{2}\left[5 + \frac{5}{2}\right] = \boxed{\frac{15}{4}}$$

(b) Find an expression for R_n. Express your answer using sigma notation and fully simplify your result.

Solution:

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$R_n = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2}{n}\cdot i\right)^2 = \frac{2}{n}\cdot\frac{2^2}{n^2}\sum_{i=1}^{n} i^2 = \boxed{\frac{8}{n^3}\sum_{i=1}^{n} i^2}$$

(c) Determine the exact value of A by taking the appropriate limit of your result from part (b). Simplify your answer fully. (No credit will be earned by using the Evaluation Theorem here.)

Solution:

$$A = \lim_{n\to\infty}\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty}\frac{4(n+1)(2n+1)}{3n^2} = \frac{4}{3}\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2}$$

$$A = \left(\frac{4}{3}\right)(2) = \boxed{\frac{8}{3}}$$

2. (23 pts) Parts (a) and (b) are unrelated.

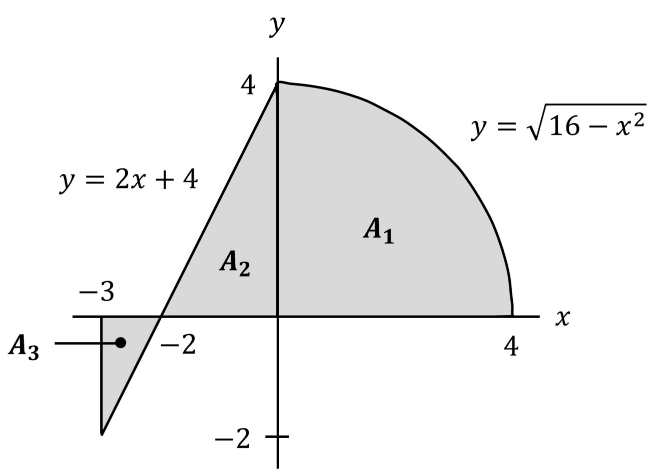
(a) Evaluate $\int_{-3}^{4} f(x)dx$, where f is defined as:

$$f(x) = \begin{cases} 2x+4 & , \quad -3 \leq x \leq 0 \\ \sqrt{16-x^2} & , \quad 0 < x \leq 4 \end{cases}$$

Fully simplify your answer. (*Hint:* Consider the relationship between definite integrals and areas.)

Solution:

The given definite integral can be interpreted in geometrical terms. Specifically, $\int_{-3}^{4} f(x)dx = A_1 + A_2 - A_3$, where A_1, A_2, and A_3 represent the shaded areas labeled in the figure below.



The region associated with A_1 is a quarter-circle of radius 4, so that $A_1 = \frac{1}{4}[\pi(4)^2] = 4\pi$.

The regions associated with A_2 and A_3 are triangles of areas $A_2 = \frac{1}{2}(2)(4) = 4$ and $A_3 = \frac{1}{2}(1)(2) = 1$, respectively.

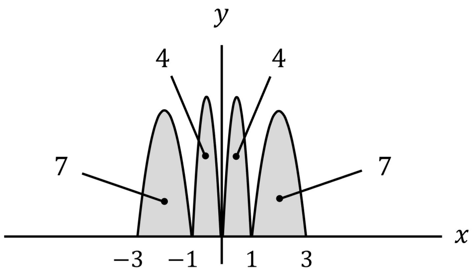
Therefore, $\int_{-3}^{4} f(x)dx = A_1 + A_2 - A_3 = 4\pi + 4 - 1 = \boxed{4\pi + 3}$

(b) Let g be a continuous function on $[-3,3]$ with $\int_{-1}^{0} g(x)dx = 4$ and $\int_{0}^{3} g(x)dx = 11$.

Find the value of $\int_{-3}^{-1} g(x)dx$ in each of the following cases:

i. g is an even function

The function depicted below is one example of a function that satisfies the given criteria.



Since g is even and $\int_{-1}^{0} g(x)dx = 4$, we know that $\int_{0}^{1} g(x)dx = 4$.

$\int_{0}^{3} g(x)dx = \int_{0}^{1} g(x)dx + \int_{1}^{3} g(x)dx$ so that $\int_{1}^{3} g(x)dx = 11 - 4 = 7$.

Since g is even and $\int_{1}^{3} g(x)dx = 7$, we know that $\int_{-3}^{-1} g(x)dx = \boxed{7}$

ii. g is an odd function

The function depicted below is one example of a function that satisfies the given criteria.

g is odd and $\int_{-1}^{0} g(x)dx = 4$

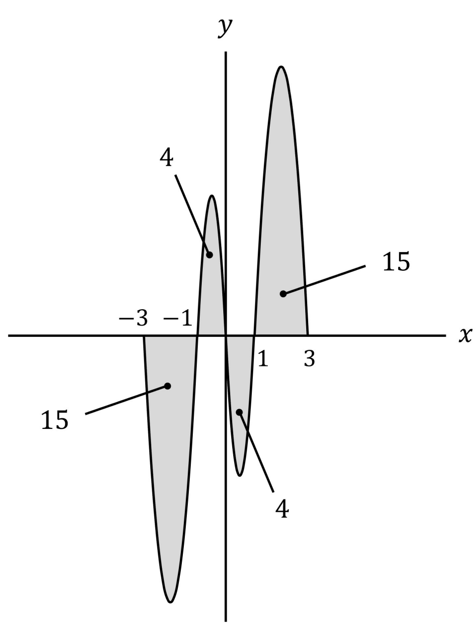
$\Rightarrow \int_{0}^{1} g(x)dx = -4$

$\int_{0}^{3} g(x)dx = \int_{0}^{1} g(x)dx + \int_{1}^{3} g(x)dx$

$\Rightarrow \int_{1}^{3} g(x)dx = 11 - (-4) = 15$

g is odd and $\int_{1}^{3} g(x)dx = 15$

$\Rightarrow \int_{-3}^{-1} g(x)dx = \boxed{-15}$



3. (24 pts) Parts (a) and (b) are unrelated.

(a) Suppose the velocity function of a car is $v(t) = 50 + 2t^3$ miles per hour between $t = 0$ hours and $t = 2$ hours. Determine the car's average velocity during that time period. Fully simplify your answer and include the correct unit of measurement.

Solution:

$$v_{\text{ave}} = \frac{1}{2-0}\int_0^2 (50 + 2t^3)dt = \frac{1}{2}\left[50t + 2 \cdot \frac{t^4}{4}\right]\Big|_0^2$$

$$= \frac{1}{2}\left[(50)(2) + \frac{2^4}{2}\right] = \frac{1}{2}[100 + 8] = \boxed{54 \text{ miles per hour}}$$

(b) Evaluate the following derivatives. Fully simplify your answers.

i. $\dfrac{d}{dx}\displaystyle\int_x^9 \sqrt{1+t^4}\, dt$

Solution:

$$\frac{d}{dx}\int_x^9 \sqrt{1+t^4}\, dt = -\frac{d}{dx}\int_9^x \sqrt{1+t^4}\, dt$$

Part 1 of the Fundamental Theorem of Calculus indicates that $\dfrac{d}{dx}\displaystyle\int_x^9 \sqrt{1+t^4}\, dt = \boxed{-\sqrt{1+x^4}}$

ii. $\dfrac{d}{dx}\displaystyle\int_{2x}^{x^2} t\sin t\, dt$

Solution:

$$\frac{d}{dx}\int_{2x}^{x^2} t\sin t\, dt = \frac{d}{dx}\left[\int_{2x}^{a} t\sin t\, dt + \int_a^{x^2} t\sin t\, dt\right] = \frac{d}{dx}\left[\int_a^{x^2} t\sin t\, dt - \int_a^{2x} t\sin t\, dt\right]$$

(Since $t\sin t$ is continuous on $(-\infty, \infty)$, a can be any real number)

Part 1 of the Fundamental Theorem of Calculus indicates that

$$\frac{d}{dx}\int_{2x}^{x^2} t\sin t\, dt = x^2\sin(x^2)\frac{d}{dx}[x^2] - 2x\sin(2x)\frac{d}{dx}[2x]$$

$$= x^2\sin(x^2)(2x) - 2x\sin(2x)(2) = \boxed{2x^3\sin(x^2) - 4x\sin(2x)}$$

4. (32 pts) Evaluate the following integrals. Fully simplify your answers.

(a) $\displaystyle\int \frac{\sin x}{\cos^4 x}\,dx$ (Express your answer in terms of x)

Solution:

Let $u = \cos x$, which implies that $du = -\sin x dx$.

$$\int \frac{\sin x}{\cos^4 x}\,dx = -\int \frac{du}{u^4} = -\int u^{-4}\,du = -\left[\frac{u^{-3}}{-3}\right] + C = \frac{1}{3u^3} + C$$

$$\int \frac{\sin x}{\cos^4 x}\,dx = \frac{1}{3\cos^3 x} + C = \boxed{\frac{1}{3}\sec^3 x + C}$$

(b) $\displaystyle\int_0^2 9x^2\sqrt{2x^3+1}\,dx$

Solution:

Let $u = 2x^3 + 1$, which implies that $du = 6x^2 dx$.

When $x = 0$, $u = (2)(0)^3 + 1 = 1$ and when $x = 2$, $u = (2)(2)^3 + 1 = 17$.

$$\int_0^2 9x^2\sqrt{2x^3+1}\,dx = \frac{9}{6}\int_0^2 \sqrt{2x^3+1}\cdot(6x^2\,dx) = \frac{3}{2}\int_1^{17} u^{1/2}du$$

$$= \left(\frac{3}{2}\right)\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_1^{17} = 17^{3/2} - 1^{3/2} = \boxed{17\sqrt{17} - 1}$$

(c) $\int x(x-3)^{1/5}\,dx$ (Express your answer in terms of x)

Solution:

Let $u = x - 3$, which implies that $du = dx$ and $x = u + 3$.

$$\begin{aligned}\int x(x-3)^{1/5}\,dx &= \int (u+3)u^{1/5}\,du = \int (u^{6/5} + 3u^{1/5})\,du \\ &= \frac{u^{11/5}}{11/5} + 3\cdot\frac{u^{6/5}}{6/5} + C = \frac{5}{11}\,u^{11/5} + 3\cdot\frac{5}{6}\,u^{6/5} + C \\ &= \boxed{\frac{5}{11}\,(x-3)^{11/5} + \frac{5}{2}\,(x-3)^{6/5} + C}\end{aligned}$$

END OF TEST

Your Initials _________

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.