APPM 1345
Exam 1
Spring 2023

Name	
Student ID	
Instructor Richard McNamara	**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

1. (26 pts) Parts (a) and (b) are unrelated.

   (a) Find the most general form of $u(x)$ such that $u'(x) = \sec^2 x + (2 - \sqrt{x})^2$.

   Solution:

   $$u'(x) = \sec^2 x + (4 - 4\sqrt{x} + x) = \sec^2 x + 4 - 4x^{1/2} + x$$

   $$u(x) = \tan x + 4x - (4)\left(\frac{2}{3}x^{3/2}\right) + \frac{x^2}{2} + C$$

   $$u(x) = \boxed{\tan x + 4x - \frac{8}{3}x^{3/2} + \frac{x^2}{2} + C}$$

   (b) Suppose the acceleration function of a particle is given by $a(t) = 4\cos t - 3\sin t + 5t$, and the particle's initial velocity and position are $v(0) = 4$ and $s(0) = 5$, respectively. Find the particle's position function $s(t)$.

Solution:

$$v(t) = 4\sin t + 3\cos t + \frac{5t^2}{2} + C_1$$

$$v(0) = 4 = 4\sin(0) + 3\cos(0) + \frac{(5)(0)^2}{2} + C_1 = 0 + 3 + 0 + C_1 = 3 + C_1 \;\Rightarrow\; C_1 = 1$$

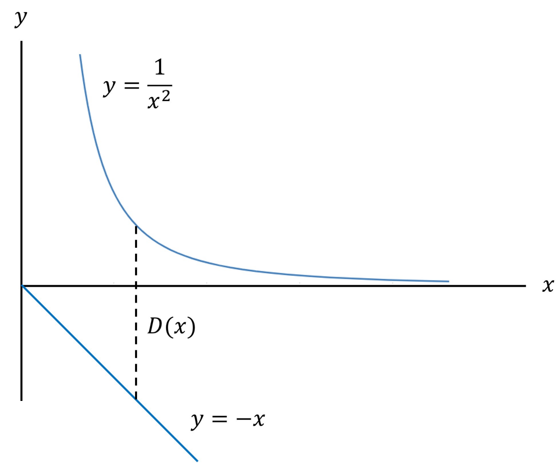
$$v(t) = 4\sin t + 3\cos t + \frac{5t^2}{2} + 1$$

$$s(t) = -4\cos t + 3\sin t + \frac{5t^3}{6} + t + C_2$$

$$s(0) = 5 = -4\cos(0) + 3\sin(0) + \frac{(5)(0)^3}{6} + 0 + C_2 = -4 + 0 + 0 + 0 + C_2 = -4 + C_2 \;\Rightarrow\; C_2 = 9$$

$$s(t) = \boxed{-4\cos t + 3\sin t + \frac{5t^3}{6} + t + 9}$$

2. (13 pts) Let the function $D(x)$ represent the vertical distance between the curve $y = 1/x^2$ and the line $y = -x$ at a given value of x. What is the minimum possible value of $D(x)$ on the interval $(0, \infty)$? Use the Second Derivative Test to confirm that the result is indeed a local minimum value of the function $D(x)$.



Solution:

$D(x) = 1/x^2 - (-x) = x^{-2} + x$

$D'(x) = -2x^{-3} + 1 = x^{-3}(-2 + x^3) = \dfrac{x^3 - 2}{x^3}$

$D' = 0$ for $x^3 = 2$, which occurs at the critical number $x = 2^{1/3}$.

Although D' does not exist at $x = 0$, $D(0)$ is undefined. Therefore, $x = 0$ is not a critical number of D.

$D''(x) = 6x^{-4} = \dfrac{6}{x^4} > 0$ for all x in the domain of D, and notably at the critical number $x = 2^{1/3}$.

Therefore, the Second Derivative Test indicates that $D(x)$ has a local minimum value at $x = 2^{1/3}$. To find the actual minimum distance, evaluate $D(2^{1/3})$, as follows:

$$D(2^{1/3}) = (2^{1/3})^{-2} + 2^{1/3} = 2^{-2/3} + 2^{1/3} = 2^{-2/3}(1 + 2) = \boxed{(3)(2^{-2/3}) = \frac{3}{\sqrt[3]{4}}}$$

3. (17 pts) Suppose Newton's Method is used to estimate the value of a root of $y = p(x) = x^3 - 5x$ using an initial estimate of $x_0 = 1$.

   (a) Write the expression for Newton's Method for the specified function $p(x)$. Your answer should be an expression for x_{n+1} in terms of x_n.

   (b) Use the expression from part (a) to determine the values of x_1 and x_2.

   (c) Would Newton's Method converge to a solution in this case? Explain why or why not.

Solution:

(a) The general expression for Newton's Method is $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)}$.

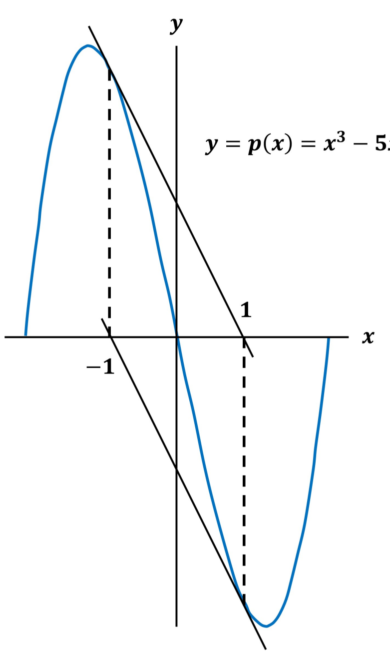
Since $p'(x) = 3x^2 - 5$, the following expression represents Newton's Method for $p(x) = x^3 - 5x$:

$$\boxed{x_{n+1} = x_n - \frac{x_n^3 - 5x_n}{3x_n^2 - 5}}$$

(b) $$x_1 = x_0 - \frac{x_0^3 - 5x_0}{3x_0^2 - 5} = 1 - \frac{1^3 - (5)(1)}{(3)(1)^2 - 5} = 1 - \left(\frac{-4}{-2}\right) \Rightarrow \boxed{x_1 = -1}$$

$$x_2 = x_1 - \frac{x_1^3 - 5x_1}{3x_1^2 - 5} = -1 - \frac{(-1)^3 - (5)(-1)}{(3)(-1)^2 - 5} = -1 - \left(\frac{4}{-2}\right) \Rightarrow \boxed{x_2 = 1}$$

(c) Newton's Method would $\boxed{\text{not converge}}$ because an endless cycle is produced. Specifically, the sequence of estimates would toggle endlessly between 1 and -1.



4. Parts (a) and (b) are not related.

   (a) (22 pts) Let $f(x) = x^{1/3} - x^{4/3}$.

      i. Identify all critical numbers of $f(x)$.

      ii. For which values of x is $f(x)$ increasing and for which values of x is $f(x)$ decreasing? Express your answers using interval notation.

      iii. Identify the x-coordinate of each local maximum and minimum of $f(x)$ (if any). Use the First Derivative Test to classify each one.

Solution:

i. $f'(x) = \frac{1}{3}x^{-2/3} - \frac{4}{3}x^{1/3} = \frac{1}{3}x^{-2/3}(1-4x) = \frac{1}{3\sqrt[3]{x^2}}(1-4x)$

$f' = 0$ at $x = 1/4$

f' does not exist at $x = 0$, and $x = 0$ is in the domain of f

Therefore, f has two critical numbers: $\boxed{0, 1/4}$

ii. $\frac{1}{\sqrt[3]{x^2}} > 0$ for all $x \neq 0$

$x < 1/4$: $(1-4x) > 0$

$x > 1/4$: $(1-4x) < 0$

It follows that $f' > 0$ on $(-\infty, 0) \cup (0, 1/4)$ and $f' < 0$ on $(1/4, \infty)$.

Since f is continuous at $x = 0$ and $f' > 0$ immediately to the left and right of $x = 0$, f is increasing at $x = 0$ despite the nonexistence of $f'(0)$.

Therefore, $\boxed{f \text{ is increasing on } (-\infty, 1/4)}$ and $\boxed{f \text{ is decreasing on } (1/4, \infty)}$

iii. f is continuous at $x = 1/4$ and f transitions from increasing to decreasing at that location. Therefore,

$\boxed{f \text{ has a local maximum at } x = 1/4}$

Since f' does not change sign at $x = 0$, f has no local maximum or minimum at $x = 0$.

(b) (22 pts) Let $g(x) = 3x^5 - 5x^4$.

i. For which values of x is $g(x)$ concave up and for which values of x is $g(x)$ concave down? Express your answers using interval notation.

ii. Identify the x-coordinate of each inflection point of $g(x)$ (if any). Justify your answer.

Solution:

i. $g'(x) = 15x^4 - 20x^3$

$g''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$

$g'' = 0$ at $x = 0, 1$

$x^2 > 0$ for all $x \neq 0$

$x < 1$: $(x-1) < 0$

$x > 1$: $(x-1) > 0$

It follows that $g'' < 0$ on $(-\infty, 0) \cup (0, 1)$ and $g'' > 0$ on $(1, \infty)$.

Therefore, $\boxed{g \text{ is concave down on } (-\infty, 0) \cup (0, 1)}$ and $\boxed{g \text{ is concave up on } (1, \infty)}$

ii. Note that g is continuous on $(-\infty, \infty)$, and in particular at $x = 0$ and $x = 1$.

g does not change concavity at $x = 0$, so there is no inflection point at $x = 0$.

g does change concavity at $x = 1$, so there is an $\boxed{\text{inflection point at } x = 1}$

END OF TEST

Your Initials _________

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.