1. (24pts) The following *parts* of this **problem** are not related.

(a)(12pts) Suppose the function $g(x) = \dfrac{x+2}{x-3}$ is *one-to-one*, find the inverse $g^{-1}(x)$. Show all work.

(b)(12pts) If f is a one-to-one function with $f(0) = 7$ and $f'(0) = 3$, find $\dfrac{d}{dx}f^{-1}(7)$ given $[f^{-1}(a)]' = [f'(f^{-1}(a))]^{-1}$.

Solution: (a)(12pts) Solving for x gives

$$y = \frac{x+2}{x-3} \Rightarrow y(x-3) = x+2 \Rightarrow xy - 3y = x+2$$

$$\Rightarrow xy - x = 3y+2 \Rightarrow x(y-1) = 3y+2 \Rightarrow x = \frac{3y+2}{y-1} \Rightarrow \boxed{f^{-1}(x) = \frac{3x+2}{x-1}.}$$

(b)(12pts) Note that $f(0) = 7 \Rightarrow f^{-1}(7) = 0$ thus

$$\frac{d}{dx}f^{-1}(7) = \left[f'(f^{-1}(7))\right]^{-1} = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(0)} = \frac{1}{3} \Rightarrow \boxed{\frac{d}{dx}f^{-1}(7) = \frac{1}{3}.}$$

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2. (28pts) Start this **problem** on a new page. The following *parts* are not related.

(a)(12pts) A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420. Write down the solution of the differential equation $\dfrac{dy}{dt} = ky$, $y(0) = y_0$ (no justification necessary for the solution of the DE) and then find the *relative growth rate*, k, of the bacteria population based on the given information.

(b)(12pts) Use the *Product Rule* to find the derivative of the function $f(x) = \sin(x)\ln(x^2+1)$.

(c)(4pts) *Multiple Choice:* If we use the following definition of the derivative: $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ to evaluate the limit $\lim_{x\to 0} \dfrac{\ln(1+2x)}{x}$ then which choice below do we get?

(**No justification necessary, choose only one answer** – *copy down the entire answer in your bluebook.*)

(A) 0 (B) $\dfrac{0}{0}$ (C) $\dfrac{1}{2}$ (D) 1 (E) 2

Solution: (a)(12pts) Note that, by a theorem in class, we have $\dfrac{dy}{dt} = ky \Rightarrow y = y_0e^{kt}$ and we are given that $y(0) = 100 = y_0$ so $y = 100e^{kt}$ and

$$420 = 100e^k \Rightarrow e^k = \frac{420}{100} \Rightarrow \ln(e^k) = \ln(4.2) \Rightarrow \boxed{\text{Growth rate is } k = \ln(4.2).}$$

(b)(12pts) The Product Rule (with some input from Chain Rule) doth sayeth:

$$f'(x) = \left[\sin(x)\ln(x^2+1)\right]' = \cos(x)\ln(x^2+1) + \sin(x)\cdot\left(\frac{1}{x^2+1}\cdot 2x\right) = \boxed{\cos(x)\ln(x^2+1) + \frac{2x\sin(x)}{x^2+1}.}$$

(c)(4pts) $\boxed{\text{Choice (E)}}$ *Discussion:* If we let $f(x) = \ln(1+2x)$ and $a = 0$ then we have

$$\lim_{x\to 0} \frac{\ln(1+2x)}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0) = \frac{d}{dx}\left[\ln(1+2x)\right]\Big|_{x=0} = \frac{2}{1+2x}\Big|_{x=0} = 2 \Rightarrow \text{ Choice (E).}$$

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3. (24pts) Start this **problem** on a new page. The following *parts* are not related.

(a)(12pts) Use the *Quotient Rule* to find the $f'(x)$ if $f(x) = \dfrac{e^x}{1+e^x}$. Simplify your answer.

(b)(12pts) Use *logarithmic differentiation* to find the derivative of: $\dfrac{(x+1)^4}{(x-3)^8}$.

Solution: (a)(12pts) Hear ye, in accordance with the Quotient Rule, we declare

$$\left[\frac{e^x}{1+e^x}\right]' = \frac{e^x(1+e^x) - e^x e^x}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \boxed{\frac{e^x}{(1+e^x)^2}.}$$

(b)(12pts) Let $y = \frac{(x+1)^4}{(x-3)^8}$ then taking the natural log of both sides and differentiating with respect to x yields

$$y = \frac{(x+1)^4}{(x-3)^8} \Rightarrow \ln(y) = \ln\left[\frac{(x+1)^4}{(x-3)^8}\right] = 4\ln(x+1) - 8\ln(x-3)$$

$$\overset{d/dx}{\Rightarrow} \frac{y'}{y} = \frac{4}{x+1} - \frac{8}{x-3}$$

$$\Rightarrow y' = y\left[\frac{4}{x+1} - \frac{8}{x-3}\right] \Rightarrow \boxed{y' = \frac{(x+1)^4}{(x-3)^8}\left[\frac{4}{x+1} - \frac{8}{x-3}\right]} = -\frac{4(x+5)(x+1)^3}{(x-3)^9}.$$

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4. (24pts) Start this **problem** on a new page. The following *parts* are not related.

(a)(10pts) Use u-substitution to find the antiderivative: $\displaystyle\int \frac{\ln(1+2x)}{2x+1}\,dx$.

(b)(10pts) Evaluate the definite integral: $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{e^{1/x}}{x^2}\,dx$.

(c)(4pts) *Multiple Choice:* The *horizontal asymptotes* of the function $f(x) = \dfrac{e^{2x} - e^x}{e^{2x}+1}$ are given by which choice below?

(**No justification necessary, choose only one answer** – *copy down the entire answer in your bluebook.*)

(A) $y = 0, 1$ (B) $y = \pm 1$ (C) $y = 0, \frac{1}{2}$ (D) $y = \frac{1}{e^2}, 1$ (E) $y = \frac{1}{e}, 1$

Solution: (a)(10pts) Using the u-substitution $u = \ln(1+2x) \Rightarrow du = \dfrac{2}{1+2x}dx \Rightarrow \dfrac{du}{2} = \dfrac{dx}{2x+1}$ thus

$$\int \frac{\ln(1+2x)}{2x+1}\,dx = \int \ln(1+2x)\,\frac{dx}{2x+1} = \frac{1}{2}\int u\,du = \frac{1}{2}\cdot\frac{u^2}{2} + C = \boxed{\frac{1}{4}\ln^2(1+2x) + C.}$$

(b)(10pts) If we let $u = \dfrac{1}{x} \Rightarrow du = -\dfrac{1}{x^2}dx$ and $x = \dfrac{1}{4} \Rightarrow u = 4$ and $x = \dfrac{1}{2} \Rightarrow u = 2$ and so

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{e^{1/x}}{x^2}\,dx = -\int_4^2 e^u\,du = \int_2^4 e^u\,du = e^u\Big|_2^4 = e^4 - e^2 = \boxed{e^2(e^2-1).}$$

(c)(4pts) Choice A. *Discussion:* Note that $e^{2x} \to \infty$ as $x \to \infty$ and $e^{2x} \to 0$ as $x \to -\infty$ thus taking limits yields

$$\lim_{x\to\infty} \frac{e^{2x} - e^x}{e^{2x} + 1} \overset{DOP}{\approx} \lim_{x\to\infty} \frac{e^{2x}}{e^{2x}} = 1 \text{ and } \lim_{x\to-\infty} \frac{e^{2x} - e^x}{e^{2x} + 1} = \frac{0-0}{0+1} = 0 \Rightarrow \text{HAs: } y = 0, 1 \Rightarrow \text{Choice (A).}$$