1. (24pts) The following parts of this problem are not related.

(a)(12pts) Suppose the function \( g(x) = \frac{x + 2}{x - 3} \) is one-to-one, find the inverse \( g^{-1}(x) \). Show all work.

(b)(12pts) If \( f \) is a one-to-one function with \( f(0) = 7 \) and \( f'(0) = 3 \), find \( \frac{d}{dx} f^{-1}(7) \) given 
\[
[f^{-1}(a)]' = \frac{1}{f'(f^{-1}(a))}.
\]

Solution: (a)(12pts) Solving for \( x \) gives
\[
y = \frac{x + 2}{x - 3} \Rightarrow y(x - 3) = x + 2 \Rightarrow xy - 3y = x + 2
\]
\[
\Rightarrow xy - x = 3y + 2 \Rightarrow x(y - 1) = 3y + 2 \Rightarrow x = \frac{3y + 2}{y - 1} \Rightarrow f^{-1}(x) = \frac{3x + 2}{x - 1}.
\]

(b)(12pts) Note that \( f(0) = 7 \Rightarrow f^{-1}(7) = 0 \) thus
\[
\frac{d}{dx} f^{-1}(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(0)} = \frac{1}{3} \Rightarrow \frac{d}{dx} f^{-1}(7) = \frac{1}{3}.
\]

2. (28pts) Start this problem on a new page. The following parts are not related.

(a)(12pts) A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420. Write down the solution of the differential equation \( \frac{dy}{dt} = ky \), \( y(0) = y_0 \) (no justification necessary for the solution of the DE) and then find the relative growth rate, \( k \), of the bacteria population based on the given information.

(b)(12pts) Use the Product Rule to find the derivative of the function \( f(x) = \sin(x) \ln(x^2 + 1) \).

(c)(4pts) **Multiple Choice:** If we use the following definition of the derivative: \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) to evaluate the limit \( \lim_{x \to 0} \frac{\ln(1 + 2x)}{x} \) then which choice below do we get?

(No justification necessary, choose only one answer – copy down the entire answer in your bluebook.)

(A) 0   (B) \( \frac{0}{0} \)   (C) \( \frac{1}{2} \)   (D) 1   (E) 2

Solution: (a)(12pts) Note that, by a theorem in class, we have \( \frac{dy}{dt} = ky \Rightarrow y = y_0 e^{kt} \) and we are given that \( y(0) = 100 = y_0 \) so \( y = 100e^{kt} \) and
\[
420 = 100e^k \Rightarrow e^k = \frac{420}{100} \Rightarrow \ln(e^k) = \ln(4.2) \Rightarrow \text{Growth rate is } k = \ln(4.2).
\]

(b)(12pts) The Product Rule (with some input from Chain Rule) doth sayeth:
\[
f'(x) = \left[\sin(x) \ln(x^2 + 1)\right]' = \cos(x) \ln(x^2 + 1) + \sin(x) \cdot \left(\frac{1}{x^2 + 1} \cdot 2x\right) = \cos(x) \ln(x^2 + 1) + \frac{2x \sin(x)}{x^2 + 1}.
\]
3. (24pts) Start this problem on a new page. The following parts are not related.

(a)(12pts) Use the Quotient Rule to find the \( f'(x) \) if \( f(x) = \frac{e^x}{1+e^x} \). Simplify your answer.

(b)(12pts) Use logarithmic differentiation to find the derivative of: \( \frac{(x+1)^4}{(x-3)^8} \).

**Solution:** (a)(12pts) Hear ye, in accordance with the Quotient Rule, we declare

\[
\left[ \frac{e^x}{1+e^x} \right]' = \frac{e^x(1+e^x) - e^x e^x}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}.
\]

(b)(12pts) Let \( y = \frac{(x+1)^4}{(x-3)^8} \) then taking the natural log of both sides and differentiating with respect to \( x \) yields

\[
y = \frac{(x+1)^4}{(x-3)^8} \Rightarrow \ln(y) = \ln \left[ \frac{(x+1)^4}{(x-3)^8} \right] = 4 \ln(x+1) - 8 \ln(x-3)
\]

\[
d/dx \frac{y'}{y} = \frac{4}{x+1} - \frac{8}{x-3} \Rightarrow y' = y \left[ \frac{4}{x+1} - \frac{8}{x-3} \right] = \frac{(x+1)^4}{(x-3)^8} \left[ \frac{4}{x+1} - \frac{8}{x-3} \right] = -\frac{4(x+5)(x+1)^3}{(x-3)^9}.
\]

4. (24pts) Start this problem on a new page. The following parts are not related.

(a)(10pts) Use \( u \)-substitution to find the antiderivative: \( \int \frac{\ln(1+2x)}{2x+1} \, dx \).

(b)(10pts) Evaluate the definite integral: \( \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{e^{1/x}}{x^2} \, dx \).

(c)(4pts) Multiple Choice: The horizontal asymptotes of the function \( f(x) = \frac{e^{2x} - e^x}{e^{2x} + 1} \) are given by which choice below?

(No justification necessary, choose only one answer – copy down the entire answer in your bluebook.)

(A) \( y = 0, 1 \)  \hspace{1cm} (B) \( y = \pm 1 \)  \hspace{1cm} (C) \( y = 0, \frac{1}{2} \)  \hspace{1cm} (D) \( y = \frac{1}{e^2}, 1 \)  \hspace{1cm} (E) \( y = \frac{1}{e}, 1 \)

**Solution:** (a)(10pts) Using the \( u \)-substitution \( u = \ln(1+2x) \Rightarrow du = \frac{2}{1+2x} \, dx \Rightarrow \frac{du}{2} = \frac{dx}{2x+1} \) thus

\[
\int \frac{\ln(1+2x)}{2x+1} \, dx = \int \ln(1+2x) \cdot \frac{dx}{2x+1} = \frac{1}{2} \int u \, du = \frac{1}{2} \cdot \frac{u^2}{2} + C = \frac{1}{4} \ln^2(1+2x) + C.
\]

(b)(10pts) If we let \( u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} \, dx \) and \( x = \frac{1}{4} \Rightarrow u = 4 \) and \( x = \frac{1}{2} \Rightarrow u = 2 \) and so

\[
\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{e^{1/x}}{x^2} \, dx = -\int_{4}^{2} e^u \, du = \int_{2}^{4} e^u \, du = e^4 \bigg|_{2} = e^4 - e^2 = e^2(e^2-1).
\]
(c)(4pts) \textbf{Choice A.} \textit{Discussion:} Note that $e^{2x} \to \infty$ as $x \to \infty$ and $e^{2x} \to 0$ as $x \to -\infty$ thus taking limits yields

$$\lim_{x \to \infty} \frac{e^{2x} - e^x}{e^{2x} + 1} \approx \lim_{x \to \infty} \frac{e^{2x}}{e^{2x}} = 1 \quad \text{and} \quad \lim_{x \to -\infty} \frac{e^{2x} - e^x}{e^{2x} + 1} = \frac{0 - 0}{0 + 1} = 0 \Rightarrow \text{HAs: } y = 0, 1 \Rightarrow \text{Choice (A).}$$