1. (24pts) The following problems are not related.

(a)(12pts) Find the value of the sum $\sum_{i=1}^{n} \frac{1}{n}\left[\frac{i}{n}+\frac{i^2}{n^2}\right]$ in terms of n. (Do **not** take any limits.). You may or may not find the following formulas useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \text{and} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

(b)(12pts) Use the Fundamental Theorem of Calculus to evaluate the integral: $\int_0^2 (2-x^2)\, dx$

Solution: (a)(12pts) Note that

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{n}\left[\frac{i}{n}+\frac{i^2}{n^2}\right] &= \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{1}{n^2}\cdot\frac{n(n+1)}{2} + \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \boxed{\frac{n(n+1)}{2n^2} + \frac{n(n+1)(2n+1)}{6n^3}.}\end{aligned}$$

(b)(12pts) Note that, by the Fundamental Theorem of Calculus, we have

$$\int_0^2 (2-x^2)\, dx = 2x - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \boxed{\frac{4}{3}.}$$

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2. (28pts) Start this problem on a **new** page. The following problems are not related.

(a)(12pts) Approximate the area under the curve $y = x^2+2x+4$ from $x=0$ to $x=6$ with a *Riemann sum* using $n=3$ subintervals of equal width and left endpoints (that is, find the approximation L_3).

(b)(12pts) Write the expression $\int_2^5 f(x)\,dx + \int_{-2}^{2} f(t)\,dt - \int_{-2}^{-1} f(x)\,dx$ as a single integral in the form $\int_a^b f(x)\,dx$.

(c)(4pts) *(Multiple Choice)* Using right endpoints (R_n) and subintervals of equal width, which limit below is equal to the definite integral $\int_1^3 \frac{x}{x^2+4}\,dx$? (**No justification necessary** - *Choose only one answer, copy down the entire answer.*)

(A) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i/n}{(2i/n)^2+4}\cdot\frac{2}{n}$ (B) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1+2i/n}{(1+2i/n)^2+4}\cdot\frac{2}{n}$ (C) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1+2(i-1)/n}{(1+2(i-1)/n)^2+4}\cdot\frac{2i}{n}$ (D) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i/n}{(2i/n)^2+4}$

Solution:

(a)(12pts) Note that $\Delta x = \frac{6-0}{3} = 2$ implies that $x_0 = 0$, $x_1 = 2$, $x_2 = 4$ and $x_3 = 6$, thus, using left endpoints yields the approximation

$$\int_0^6 [x^2+2x+4]\,dx \approx L_3 = \sum_{i=1}^{3} f(x_{i-1})\Delta x = f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2 = 2\,[4+12+28] = 2\cdot 44 = \boxed{88.}$$

(b)(12pts) Note that in a definite integral the variable of integration is a *dummy variable* and can be replaced with any other variable so we can write $\int_{-2}^{2} f(t)\,dt = \int_{-2}^{2} f(x)\,dx$ thus

$$\int_2^5 f(x)\,dx + \int_{-2}^{2} f(t)\,dt - \int_{-2}^{-1} f(x)\,dx = \int_{-1}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_2^5 f(x)\,dx = \boxed{\int_{-1}^{5} f(x)\,dx.}$$

(c)(4pts) Choice (B). *Discussion:* We know that $a = 1$ and $b = 3$ from the integral, so $\Delta x = (3-1)/n = 2/n$. Likewise, using right endpoints, we find that $x_i = 1 + 2i/n$. Combining this information into the limit definition, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1+2i/n}{(1+2i/n)^2+4}\,\frac{2}{n} \;\Rightarrow\; \text{Choice (B).}$$

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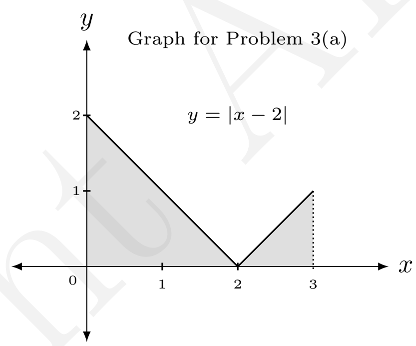
3. (24pts) Start this problem on a **new** page. The following problems are not related.

(a)(12pts) Evaluate the definite integral $\displaystyle\int_0^3 |x-2|\,dx$.

(b)(12pts) Use a u-substitution to evaluate the indefinite integral $\displaystyle\int x\sqrt{x-1}dx$. Show all work.

Solution: (a)(12pts) This problem can be done geometrically or directly. Note that the graph of $f(x) = |x-2|$, where $0 \le x \le 3$, looks like:



and so we see the region of interest can be described by two triangles, one triangle with base and height length of 2 units and the other triangle with base and height length of 1 unit, thus

$$\int_0^3 |x-2|\,dx = \frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}\cdot 1\cdot 1 = \frac{4}{2} + \frac{1}{2} = \boxed{\frac{5}{2}.}$$

We can also do the integral directly by applying the definition of the absolute value and separating the integral:

$$\int_0^3 |x-2|\,dx = \int_0^2 |x-2|\,dx + \int_2^3 |x-2|\,dx = \int_0^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx$$

thus,

$$\begin{aligned}
\int_0^3 |x-2|\,dx &= \int_0^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx \\
&= -\left(\frac{x^2}{2} - 2x\right)\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^3 \\
&= -\left(\frac{2^2}{2} - 2\cdot 2\right) + 0 + \left(\frac{3^2}{2} - 2\cdot 3\right) - \left(\frac{2^2}{2} - 2\cdot 2\right) = 2 + \frac{9}{2} - 6 + 2 = \frac{9}{2} - 2 = \boxed{\frac{5}{2}.}
\end{aligned}$$

(b)(12pts) If we use the u-substitution $u = x - 1$ then $du = dx$ and $x = u + 1$ and so

$$\int x\sqrt{x-1}\,dx = \int (u+1)\sqrt{u}\,du = \int (u^{3/2} + u^{1/2})\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}$$

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4. (24pts) Start this problem on a **new** page. The following problems are not related.

(a)(10pts) Evaluate the definite integral: $\displaystyle\int_0^{\pi/2} \sin^2(x)\cos(x)\,dx$

(b)(10pts) If $f(x) = \int_4^{x^2} \frac{t-1}{t^2+1}\, dt$, use the Fundamental Theorem of Calculus to find $f'(2)$. Simplify your answer.

(c)(4pts) Suppose we have a rectangle of width $w = 4$, what should the height, h, of the rectangle be so that the area of the rectangle and the area bounded by the curve $f(x) = \sqrt{x}$, for $0 \le x \le 4$, and the x-axis are the same?
(**No justification necessary** - *Choose only one answer, copy down the entire answer*)

(A) $h = \frac{1}{2}$ (B) $h = \frac{4}{3}$ (C) $h = \frac{1}{4}$ (D) $h = \frac{2}{3}$ (E) None of these

Solution:

(a)(10pts) We use the u-substitution $u = \sin(x) \Rightarrow du = \cos(x)\, dx$ and $x = 0 \Rightarrow u = \sin(0) = 0$ and $x = \pi/2 \Rightarrow u = \sin(\pi/2) = 1$. Thus

$$\int_0^{\pi/2} \sin^2(x)\cos(x)\, dx = \int_0^1 u^2\, du = \frac{u^3}{3}\Big|_0^1 = \boxed{\frac{1}{3}.}$$

(b)(10pts) By the Fundamental Theorem of Calculus we have

$$f'(x) = \frac{d}{dx}\left[\int_4^{x^2} \frac{t-1}{t^2+1}\, dt\right] = \frac{x^2-1}{(x^2)^2+1}\cdot 2x = \frac{2x(x^2-1)}{x^4+1} \Rightarrow f'(2) = \frac{2\cdot 2\cdot(2^2-1)}{2^4+1} = \frac{4\cdot 3}{17} = \boxed{\frac{12}{17}}$$

(c)(4pts) $\boxed{\text{Choice (B).}}$ *Discussion:* In the case that $f(x) \ge 0$, the height of the rectangle should equal the average value of $f(x)$ since

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx \;\Rightarrow\; f_{ave}\cdot(b-a) = \int_a^b f(x)\, dx = \text{Area below } f(x), \text{ for } a \le x \le b,$$

and, in this case, we have

$$f_{ave} = \frac{1}{4-0}\int_0^4 \sqrt{x}\, dx = \frac{1}{4}\left[\frac{2}{3}x^{3/2}\right]\Big|_0^4 = \frac{1}{4}\left[\frac{2}{3}\cdot 2^3\right] = \frac{1}{4}\cdot\frac{16}{3} = \frac{4}{3} \Rightarrow \text{ let } h = \frac{4}{3} \Rightarrow \text{ Choice (B).}$$