1. (24pts) The following problems are not related.

   (a)(12pts) Find all the intervals on which $g(x) = 3x^5 - 5x^3$ is *increasing* or *decreasing*. **Give your answer in interval notation.** Show all work.

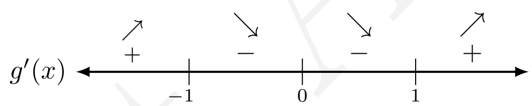
   (b)(12pts) Find and classify any critical point(s) of $f(x) = x + 2\cos(x),\ 0 \le x \le \frac{\pi}{2}$. Classify the critical point using the **2nd Derivative Test**. (You do not need to find the y-value of any critical point.)

   Solution: (a)(12pts) Since $g(x)$ is a polynomial, the only critical points are the roots of $g'(x)$ where

$$g'(x) = [3x^5 - 5x^3]' = 15x^4 - 15x^2 = 15x^2(x^2-1) = 15x^2(x-1)(x+1) \ \Rightarrow\ g'(x) = 0 \text{ only if } x = -1, 0, 1$$

   and checking (for example) the sign chart for $g'(x) = 15x^2(x-1)(x+1)$ gives

Sign chart for $g'(x)$



   thus $\boxed{g(x) \text{ decreases on } (-1,0) \cup (0,1) \text{ and increases on } (-\infty,-1) \cup (1,\infty).}$

   (b)(12pts) We first search for roots of $f'(x)$ in $(0, \pi/2)$ to find the critical points,

$$f'(x) = 0 \ \Rightarrow\ [x + 2\cos(x)]' = 0 \ \Rightarrow\ 1 - 2\sin(x) = 0 \ \Rightarrow\ \sin(x) = \frac{1}{2} \ \Rightarrow\ x = \frac{\pi}{6} \in (0, \pi/2)$$

   and, for the 2nd Derivative Test, note that

$$f''(x) = [1 - 2\sin(x)]' = -2\cos(x) \ \Rightarrow\ f''\left(\frac{\pi}{6}\right) = -2\cos\left(\frac{\pi}{6}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \ \Rightarrow\ \text{Concave down}$$

   so since $f''\left(\frac{\pi}{6}\right) < 0$, by the 2nd Derivative Test, we have $\boxed{\text{a local max at } x = \frac{\pi}{6}.}$

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2. (28pts) Start this problem on a **new** page. The following problems are not related.

   (a)(12pts) *Consider the following problem:* The monthly production of a light bulb company is $P = 4xy$ (in millions) where x is the cost of equipment and y is the cost of labor (in millions of dollars). The company needs to produce $P = 1$ million units, which values of x and y will minimize the cost $C = x + y$? **Answer the following questions:** $\boxed{(i)(4\text{pts})}$ Is this is a *minimization* or *maximization* problem? Write down a function in terms of the two variables x and y that you would minimize (or maximize). $\boxed{(ii)(4\text{pts})}$ Use the given information to write an equation that relates the variables x and y. $\boxed{(iii)(4\text{pts})}$ Now using optimization find the value of x and y that satisfy this problem. Justify your answer by classifying your critical point(s) using either the 1st or 2nd Derivative Test.

   (b)(12pts) Suppose we want to approximate the x-intercept of $f(x) = 3x^2 - 2$ using Newton's Method. What would the formula for x_{n+1} be? (To get full points for this question you must provide the explicit formula for x_{n+1} in terms of x_n, the generic formula for Newton's Method is <u>not</u> sufficient. You do *not* need to approximate the solution. **Simplify your answer.**)

   (c)(4pts) *Multiple Choice:* If $F(x) = \dfrac{x}{x^2+1}$ is an antiderivative of $f(x)$ then $f(x)$ is equal to which choice below? (**No justification necessary** - *Choose only <u>one</u> answer, copy down the entire answer.*)

(A) $f(x) = \dfrac{x^2/2}{x^3/3 + 3x}$ (B) $f(x) = \dfrac{1 - x^2}{x^4 + 2x^2 + 1}$ (C) $f(x) = \dfrac{1}{2x}$ (D) $f(x) = \dfrac{1 - x^2}{(x+1)^2}$ (E) None of these

Solution:

(a)(i)(4pts) This is a **minimization** problem and we wish to $\boxed{\text{minimimize } C = x + y.}$

(a)(ii)(4pts) The constraint is $\boxed{P = 1 \Leftrightarrow 4xy = 1}$ which implies $y = \dfrac{1}{4x}$ or $x = \dfrac{1}{4y}$.

(a)(iii)(4pts) We have

$$C = x + y = x + \frac{1}{4x} \Rightarrow \frac{dC}{dx} = 1 - \frac{1}{4x^2} = \frac{4x^2 - 1}{4x^2} \Rightarrow C'(x) = 0 \text{ or undefined only if } x = 0, \pm\frac{1}{2}$$

but clearly we need $x > 0$ so we have $x = \dfrac{1}{2}$.

Classify: Finally note that $\dfrac{d^2C}{dx} = \dfrac{2}{4x^3} = \dfrac{1}{2x^3}$ and $C''(1/2) > 0$ which implies, by the <u>2nd Derivative Test</u> that x is a (local and) absolute min (since we only have one critical point).

$$\boxed{\text{Thus } x = \frac{1}{2} \text{ and } y = \frac{1}{4x} = \frac{1}{2} \text{ will minimize the cost } C = x + y \text{ subject to } 4xy = 1.}$$

(b)(12pts) We need to approximate a solution to $f(x) = 0$. So, by Newton's Method, we have

$$f'(x) = [3x^2 - 2]' = 6x \quad \Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n^2 - 2}{6x_n} = \frac{6x_n^2 - (3x_n^2 - 2)}{6x_n} = \frac{3x_n^2 + 2}{6x_n}$$

$$\Rightarrow \boxed{x_{n+1} = \frac{3x_n^2 + 2}{6x_n}, \ n = 1, 2, \ldots.}$$

(c)(4pts) $\boxed{\text{Choice B.}}$ *Discussion:* If $F(x)$ is an antiderivative of $f(x)$ then $F'(x) = f(x)$ so, using the Quotient Rule, we have

$$f(x) = F'(x) = \frac{d}{dx}\left[\frac{x}{x^2 + 1}\right] = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{x^4 + 2x^2 + 1} \Rightarrow \text{Choice (B).}$$

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3. (24pts) Start this problem on a **new** page. The following parts of this problem are not related.

(a)(10pts) Find *any* function $F(x)$ such that $F'(x) = \dfrac{1 + x^{5/2}}{x^{1/2}}$.

(b)(10pts) Find all *inflection points* of $g(x) = \dfrac{x^5}{20} - \dfrac{x^4}{6}$. Show all work and justify. (You do not need to find the y-value of any inflection point.)

(c)(4pts) *Multiple Choice:* Which graph below *best* matches the graph of the function $f(x) = \dfrac{3x^2}{x^2 - 1}$?
(**No justification necessary** - *Choose only <u>one</u> answer, clearly indicate your answer otherwise points will be deducted.*)

Solution: (a)(10pts) Note that

$$\int \frac{1 + x^{5/2}}{x^{1/2}}\, dx = \int \left(\frac{1}{x^{1/2}} + \frac{x^{5/2}}{x^{1/2}}\right) dx = \int \left(x^{-1/2} + x^2\right) dx = \frac{x^{1/2}}{1/2} + \frac{x^3}{3} = 2x^{1/2} + \frac{x^3}{3}$$

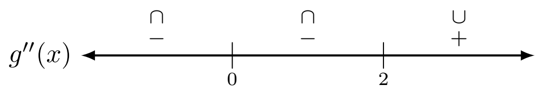
so we have $\boxed{F(x) = 2x^{1/2} + \dfrac{x^3}{3} \text{ and, more generally, } F(x) = 2x^{1/2} + \dfrac{x^3}{3} + C.}$

(b)(10pts) Since $g(x)$ is a polynomial, we only need to find and classify the roots of $g''(x)$ where

$$g'(x) = \left[\frac{x^5}{20} - \frac{x^4}{6}\right]' = \frac{5x^4}{20} - \frac{4x^3}{6} \Rightarrow g''(x) = \frac{20x^3}{20} - \frac{12x^2}{6} = x^3 - 2x^2 = x^2(x - 2).$$

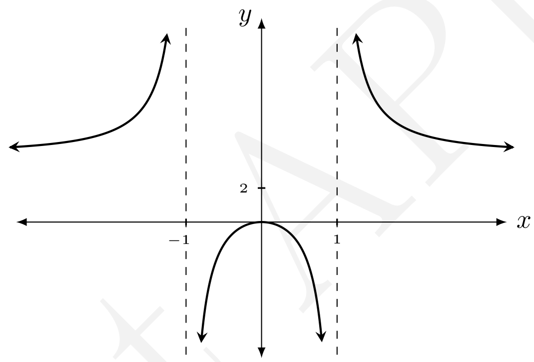
So $g''(x) = 0 \Rightarrow x = 0, 2$ and now we classify these points using (for example) a sign chart for $g''(x) = x^2(x-2)$,

Sign chart for $g''(x)$



$\Rightarrow$ Inflection Point at $(2, g(2))$.

(c)(4pts) Graph (D). <u>Discussion:</u> Note that $f(x) = \dfrac{3x^2}{x^2-1}$ has a horizontal asymptote at $y = 3$ which eliminates Graph (A), and $f(x) < 0$ for $x \in (-1, 0)$ which eliminates Graph (B), and, finally, since $f(x)$ has vertical asymptotes at $x = \pm 1$, we can eliminate Graph (C) thus the *best* match from all these choice is Graph (D).



Graph (D)

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4. (24pts) Start this problem on a **new** page. The following problems are not related.

(a)(12pts) Find the most general antiderivative of $f(t) = 2\sec(t)\tan(t) + \dfrac{1}{2t^2}$. Show all work.

(b)(12pts) Suppose the acceleration of an object at any time t is given by $a(t) = 3t^2 - 4t$ m/s^2, $t \geq 0$. Find the position, $s(t)$, at any time t if $v(1) = 1$ m/s and $s(0) = 2$. Show all work.

Solution: (a)(10pts) Note that

$$\int \left[2\sec(t)\tan(t) + \frac{1}{2t^2}\right] dt = 2\int \sec(t)\tan(t)\,dt + \frac{1}{2}\int t^{-2}\,dt = 2\sec(t) + \frac{1}{2}\cdot\frac{t^{-1}}{-1} = \boxed{2\sec(t) - \frac{1}{2t} + C.}$$

(b)(10pts) Recall that $a(t) = v'(t)$ thus

$$v(t) = \int a(t)\,dt = \int (3t^2 - 4t)\,dt = t^3 - 4\cdot\frac{t^2}{2} + C = t^3 - 2t^2 + C$$

and $v(1) = 1$ implies

$$1 = v(1) = 1^3 - 2\cdot 1^2 + C \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow v(t) = t^3 - 2t^2 + 2 = s'(t)$$
$$\Rightarrow s(t) = \frac{t^4}{4} - \frac{2}{3}t^3 + 2t + \widetilde{C}$$

and $s(0) = 2$ implies that $\widetilde{C} = 2$ so $\boxed{s(t) = \dfrac{t^4}{4} - \dfrac{2}{3}t^3 + 2t + 2.}$
