1. (24pts) The following problems are not related.

(a) (12pts) Find all the intervals on which \( g(x) = 3x^5 - 5x^3 \) is increasing or decreasing. Give your answer in interval notation. Show all work.

(b) (12pts) Find and classify any critical point(s) of \( f(x) = x + 2 \cos(x), \ 0 \leq x \leq \frac{\pi}{2} \). Classify the critical point using the 2nd Derivative Test. (You do not need to find the y-value of any critical point.)

Solution: (a) (12pts) Since \( g(x) \) is a polynomial, the only critical points are the roots of \( g'(x) \) where

\[
g'(x) = [3x^5 - 5x^3]' = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1) \Rightarrow g'(x) = 0 \text{ only if } x = -1, 0, 1
\]

and checking (for example) the sign chart for \( g'(x) = 15x^2(x - 1)(x + 1) \) gives

\[
\text{Sign chart for } g'(x)
\]

\[
g'(x) \leftarrow + \rightarrow - - \rightarrow + \]

thus \( g(x) \) decreases on \((-1, 0) \cup (0, 1)\) and increases on \((-\infty, -1) \cup (1, \infty)\).

(b) (12pts) We first search for roots of \( f'(x) \) in \((0, \pi/2)\) to find the critical points,

\[
f'(x) = 0 \Rightarrow [x + 2 \cos(x)]' = 0 \Rightarrow 1 - 2 \sin(x) = 0 \Rightarrow \sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} \in (0, \pi/2)
\]

and, for the 2nd Derivative Test, note that

\[
f''(x) = [1 - 2 \sin(x)]' = -2 \cos(x) \Rightarrow f''\left(\frac{\pi}{6}\right) = -2 \cos\left(\frac{\pi}{6}\right) = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \Rightarrow \text{Concave down}
\]

so since \( f''\left(\frac{\pi}{6}\right) < 0 \), by the 2nd Derivative Test, we have a local max at \( x = \frac{\pi}{6} \).

2. (28pts) Start this problem on a new page. The following problems are not related.

(a) (12pts) Consider the following problem: The monthly production of a light bulb company is \( P = 4xy \) (in millions) where \( x \) is the cost of equipment and \( y \) is the cost of labor (in millions of dollars). The company needs to produce \( P = 1 \) million units, which values of \( x \) and \( y \) will minimize the cost \( C = x + y \)? Answer the following questions:

(i)(4pts) Is this a minimization or maximization problem? Write down a function in terms of the two variables \( x \) and \( y \) that you would minimize (or maximize). (iii)(4pts) Use the given information to write an equation that relates the variables \( x \) and \( y \).

(iii)(4pts) Now using optimization find the value of \( x \) and \( y \) that satisfy this problem. Justify your answer by classifying your critical point(s) using either the 1st or 2nd Derivative Test.

(b) (12pts) Suppose we want to approximate the \( x \)-intercept of \( f(x) = 3x^2 - 2 \) using Newton's Method. What would the formula for \( x_{n+1} \) be? (To get full points for this question you must provide the explicit formula for \( x_{n+1} \) in terms of \( x_n \), the generic formula for Newton’s Method is not sufficient. You do not need to approximate the solution. Simplify your answer.)

(c)(4pts) Multiple Choice: If \( F(x) = \frac{x}{x^2 + 1} \) is an antiderivative of \( f(x) \) then \( f(x) \) is equal to which choice below? (No justification necessary. Choose only one answer, copy down the entire answer.)
(A) \( f(x) = \frac{x^2/2}{x^3/3 + 3x} \)  \hspace{1cm} (B) \( f(x) = \frac{1 - x^2}{x^4 + 2x^2 + 1} \)  \hspace{1cm} (C) \( f(x) = \frac{1}{2x} \)  \hspace{1cm} (D) \( f(x) = \frac{1 - x^2}{(x + 1)^2} \)  \hspace{1cm} (E) None of these

**Solution:**

(a)(i)(4pts) This is a **minimization** problem and we wish to minimize \( C = x + y \).

(a)(ii)(4pts) The constraint is \( P = 1 \Leftrightarrow 4xy = 1 \) which implies \( y = \frac{1}{4x} \) or \( x = \frac{1}{4y} \).

(a)(iii)(4pts) We have
\[
C = x + y = x + \frac{1}{4x} \Rightarrow \frac{dC}{dx} = 1 - \frac{1}{4x^2} = \frac{4x^2 - 1}{4x^2} \Rightarrow C'(x) = 0 \text{ or undefined only if } x = 0, \pm \frac{1}{2}
\]
but clearly we need \( x > 0 \) so we have \( x = \frac{1}{2} \).

**Classify:** Finally note that \( \frac{d^2C}{dx^2} = \frac{2}{4x^3} = \frac{1}{2x^3} \) and \( C''(1/2) > 0 \) which implies, by the 2nd Derivative Test that \( x \) is a (local and) absolute min (since we only have one critical point).

Thus \( x = \frac{1}{2} \) and \( y = \frac{1}{4x} = \frac{1}{2} \) will minimize the cost \( C = x + y \) subject to \( 4xy = 1 \).

(b)(12pts) We need to approximate a solution to \( f(x) = 0 \). So, by Newton’s Method, we have
\[
f'(x) = [3x^2 - 2]' = 6x \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n^2 - 2}{6x_n} = \frac{6x_n^2 - (3x_n^2 - 2)}{6x_n} = \frac{3x_n^2 + 2}{6x_n}
\]
\[
\Rightarrow x_{n+1} = \frac{3x_n^2 + 2}{6x_n}, \quad n = 1, 2, \ldots.
\]

(c)(4pts) **Choice B.** Discussion: If \( F(x) \) is an antiderivative of \( f(x) \) then \( F'(x) = f(x) \) so, using the Quotient Rule, we have
\[
f(x) = F'(x) = \frac{d}{dx} \left[ \frac{x}{x^2 + 1} \right] = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{x^4 + 2x^2 + 1} \Rightarrow \text{Choice (B)}. \]

3. (24pts) Start this problem on a **new** page. The following parts of this problem are not related.

(a)(10pts) Find any function \( F(x) \) such that \( F'(x) = \frac{1 + x^{5/2}}{x^{1/2}} \).

(b)(10pts) Find all inflection points of \( g(x) = x^5 \frac{5x^4}{20} - x^4 \frac{4x^3}{6} \). Show all work and justify. (You do not need to find the \( y \)-value of any inflection point.)

(c)(4pts) Multiple Choice: Which graph below best matches the graph of the function \( f(x) = \frac{3x^2}{x^3 - 1} \)?

**No justification necessary** - Choose only one answer, clearly indicate your answer otherwise points will be deducted.

**Solution:** (a)(10pts) Note that
\[
\int \frac{1 + x^{5/2}}{x^{1/2}} \, dx = \int \left( \frac{1}{x^{1/2}} + \frac{x^{5/2}}{x^{1/2}} \right) \, dx = \int (x^{-1/2} + x^2) \, dx = \frac{x^{1/2}}{1/2} + \frac{x^3}{3} = 2x^{1/2} + x^3/3
\]
so we have \( F(x) = 2x^{1/2} + x^3/3 \) and, more generally, \( F(x) = 2x^{1/2} + \frac{x^3}{3} + C \).

(b)(10pts) Since \( g(x) \) is a polynomial, we only need to find and classify the roots of \( g''(x) \) where
\[
g'(x) = \left[ x^5 \frac{5x^4}{20} - x^4 \frac{4x^3}{6} \right]' = 5x^4 \frac{5x^4}{20} - 4x^3 \frac{4x^3}{6} \Rightarrow g''(x) = \frac{20x^3}{20} - \frac{12x^2}{6} = x^3 - 2x^2 = x^2(x - 2).
\]
So \( g''(x) = 0 \Rightarrow x = 0, 2 \) and now we classify these points using (for example) a sign chart for \( g''(x) = x^2(x - 2) \),

Sign chart for \( g''(x) \)

\[
g''(x) \begin{cases} \begin{array}{c} 
\cap \\
0 \\
\cap \\
2 \\
\cup \\
\Rightarrow \text{Inflection Point at } (2, g(2)).
\end{array} \end{cases}
\]

(c)(4pts) \textbf{Graph (D). Discussion:} Note that \( f(x) = \frac{3x^2}{x^2 - 1} \) has a horizontal asymptote at \( y = 3 \) which eliminates Graph (A), and \( f(x) < 0 \) for \( x \in (-1, 0) \) which eliminates Graph (B), and, finally, since \( f(x) \) has vertical asymptotes at \( x = \pm 1 \), we can eliminate Graph (C) thus the \textit{best} match from all these choice is Graph (D).

![Graph (D)](image)

4. (24pts) Start this problem on a \textbf{new} page. The following problems are not related.

(a)(12pts) Find the most general antiderivative of \( f(t) = 2 \sec(t) \tan(t) + \frac{1}{2t^2} \). Show all work.

(b)(12pts) Suppose the acceleration of an object at any time \( t \) is given by \( a(t) = 3t^2 - 4t \) m/s\(^2\), \( t \geq 0 \). Find the position, \( s(t) \), at any time \( t \) if \( v(1) = 1 \) m/s and \( s(0) = 2 \). Show all work.

**Solution:** (a)(10pts) Note that

\[
\int \left[ 2 \sec(t) \tan(t) + \frac{1}{2t^2} \right] dt = 2 \int \sec(t) \tan(t) dt + \frac{1}{2} \int t^{-2} dt = 2 \sec(t) + \frac{1}{2} \cdot \frac{t^{-1}}{-1} = 2 \sec(t) - \frac{1}{2t} + C.
\]

(b)(10pts) Recall that \( a(t) = v'(t) \) thus

\[
v(t) = \int a(t) dt = \int (3t^2 - 4t) dt = t^3 - 4 \cdot \frac{t^2}{2} + C = t^3 - 2t^2 + C
\]

and \( v(1) = 1 \) implies

\[
1 = v(1) = 1^3 - 2 \cdot 1^2 + C \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow v(t) = t^3 - 2t^2 + 2 = s'(t)
\]

\[
\Rightarrow s(t) = \frac{t^4}{4} - \frac{2}{3}t^3 + 2t + \tilde{C}
\]

and \( s(0) = 2 \) implies that \( \tilde{C} = 2 \) so

\[
s(t) = \frac{t^4}{4} - \frac{2}{3}t^3 + 2t + 2.
\]