1. (24pts) The following problems are not related.

(a)(12pts) Find all the intervals on which \( f(x) = \frac{x}{x^2 + 1} \) is increasing or decreasing. Give your answer in interval notation. Show all work.

(b)(12pts) Find and classify any critical point(s) of \( f(x) = x - 2 \sin(x) \), \( 0 \leq x \leq \pi \). Classify the critical point using the 2nd Derivative Test and give your answer in the form \((x, y)\).

Solution: (a)(12pts) Using the Quotient Rule we have
\[
f'(x) = \frac{d}{dx} \left[ \frac{x}{x^2 + 1} \right] = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(1 - x)(1 + x)}{(x^2 + 1)^2} \implies f'(x) = 0 \iff x = \pm 1
\]
so \( f'(x) > 0 \) if \( x \in (-1, 1) \) and \( f'(x) < 0 \) otherwise thus
\( f(x) \) is increasing on \((-1, 1)\) and decreasing on \((-\infty, -1) \cup (1, \infty)\).

(b)(12pts) First we find the critical points, note that
\[
f'(x) = 1 - 2 \cos(x) = 0 \Rightarrow \cos(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} \in [0, \pi]
\]
and now we classify the critical point using the 2nd Derivative test. Note that
\[
f''(x) = 2 \sin(x) \Rightarrow f''(\pi/3) = 2 \sin(\pi/3) = \sqrt{3} > 0 \Rightarrow x = \pi/3 \text{ is a local minimum.}
\]
Thus there is a local minimum at \( x = \pi/3 \) with value \( f(\pi/3) = \pi/3 - \sqrt{3}, \) i.e.
\[
\text{there is a local minimum extrema point at } (\pi/3, \pi/3 - \sqrt{3}).
\]

2. (28pts) The following problems are not related.

(a)(12pts) In this problem, you will find two positive integers \( x \) and \( y \) such that the sum of the first number and four times the second number is 1000 and the product is as large as possible. Answer the following questions:

(i)(4pts) Is this a minimization or maximization problem? Write down a function in terms of the two variables \( x \) and \( y \) that you would minimize (or maximize).

(ii)(4pts) Use the given information to write an equation that relates the variables \( x \) and \( y \).

(iii)(4pts) Now using optimization find the value of \( x \) and \( y \) that satisfy this problem. Justify your answer by classifying your critical point(s) using either the 1st or 2nd Derivative Test.

(b)(12pts) Suppose we want to approximate the intersection points of \( y = x^2 \) and \( y = \cos(x) \). If we want to use Newton’s Method to do this then what would the formula for \( x_{n+1} \) be? (To get full points for this question you must provide the explicit formula for \( x_{n+1} \) in terms of \( x_n \), the generic formula for Newton’s Method is not sufficient. You do not need to approximate the point of intersection.)

(c)(4pts) Suppose the tangent line to the curve \( y = f(x) \) at the point \((2, 5)\) has the equation \( y = 9 - 2x \). If Newton’s Method is used to locate a root of the equation \( f(x) = 1 \) and the initial approximation is \( x_1 = 2 \) then the second iteration is \( x_2 = \text{see choices below (write answer in bluebook)} \). (No justification necessary - Choose only one answer, copy down the entire answer in your bluebook.)
(A) \( x_2 = \frac{2}{9} \)  \hspace{1cm} (B) \( x_2 = 4 \)  \hspace{1cm} (C) \( x_2 = 5 \)  \hspace{1cm} (D) \( x_2 = \frac{9}{2} \)  \hspace{1cm} (E) None of these.

Solution:

(a)(i)(4pts) This is a \textbf{maximization} problem and we wish to \textbf{maximize} \( P = xy \).

(a)(ii)(4pts) The constraint is \( x + 4y = 1000 \) which implies either \( y = 250 - \frac{x}{4} \) or \( x = 1000 - 4y \).

(a)(iii)(4pts) We have
\[
P = xy = x \left( 250 - \frac{x}{4} \right) = 250x - \frac{x^2}{4} \Rightarrow \frac{dP}{dx} = 250 - \frac{x}{2} \quad \text{and} \quad \frac{dP}{dx} = 0 \Rightarrow x = 500 \text{ and } y = 125
\]
or
\[
P = xy = (1000 - 4y)y = 1000y - 4y^2 \Rightarrow \frac{dP}{dy} = 1000 - 8y \quad \text{and} \quad \frac{dP}{dy} = 0 \Rightarrow y = \frac{1000}{8} = 125 \text{ and } x = 500.
\]

Classify: Finally note that, for example, \( \frac{d^2P}{dx} = -\frac{1}{2} < 0 \) (or \( \frac{d^2P}{dy} = -8 < 0 \)) which implies, by the 2nd Derivative Test that \( x = 500 \) is a (local and absolute) maximum.

(b)(12pts) We wish to approximate \( x^2 = \cos(x) \) or \( x^2 - \cos(x) = 0 \), so if we let \( f(x) = x^2 - \cos(x) \) then \( f'(x) = 2x + \sin(x) \) and so to implement Newton’s Method we use the formula
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - \cos(x_n)}{2x_n + \sin(x_n)} = \frac{x_n^2 + x_n \sin(x_n) + \cos(x_n)}{2x_n + \sin(x_n)}
\]

(c)(4pts) Choice B. Note that \( f(x) = 1 \Rightarrow f(x) - 1 = 0 \Rightarrow g(x) = 0 \) where \( g(x) = f(x) - 1 \) and since the graph of \( g(x) \) is the graph of \( f(x) \) shifted down one unit, the equation of the tangent line is also shifted down one unit, i.e. the equation of the tangent line to \( g(x) \) at \((2,5)\) is \( y = (9 - 2x) - 1 = 8 - 2x \) and the \( x \)-intercept of this tangent line is \( 8 - 2x = 0 \Rightarrow x = 4 \Rightarrow x_2 = 4 \).

3. (24pts) The following problems are not related:

(a)(12pts) For what values of the constants \( a \) and \( b \) is \((1,2)\) a \textbf{point of inflection} of the curve \( y = ax^3 + bx^2 \)?

(b)(12pts) In your blue book clearly \textbf{sketch the graph} of a function \( h(x) \) that satisfies all the following properties (label all extrema, inflection points and asymptotes):

\[
\begin{align*}
&\bullet \ h(x) \text{ is a polynomial and an odd function, } h(0) = 0 \text{ and } h(1) = 2, \\
&\bullet \ \lim_{x \to -0.5^+} h(x) = -1.5, \ \lim_{x \to -\infty} h(x) = +\infty, \text{ and } \lim_{x \to \infty} h(x) = -\infty, \\
&\bullet \ \text{The 1st derivative test for } h(x) \text{ is shown below:}
\end{align*}
\]

\[
\begin{array}{c|ccc|c}
\text{Sign chart for } h'(x) & - & + & - \\
\hline
-\frac{1}{2} & & & \\
\end{array}
\]

\[
\bullet \ h''(x) < 0 \text{ on } (-1/2,0) \text{ and } (1/2,\infty).
\]

\textbf{Solution:} (a)(12pts) First note that \( 2 = a \cdot 1^3 + b \cdot 1^2 \Rightarrow 2 = a + b \). Now note that
\[
y = ax^3 + bx^2 \Rightarrow y' = 3ax^2 + 2bx \Rightarrow y'' = 6ax + 2b
\]
and note that $y'' = 0$ when $x = 1$ (since we have an inflection point when $x = 1$) thus we have $0 = 6a + 2b$ and so we have the following system of equations

\[
\begin{align*}
a + b &= 2 \\
6a + 2b &= 0
\end{align*}
\]

\[
\Rightarrow \quad b = 2 - a \\
\Rightarrow \quad b = 3 \\
\Rightarrow \quad a = -1
\]

\[
y = -x^3 + 3x^2
\]

4. (24pts) The following problems are not related.

(a) (10pts) Find the most general antiderivative of the function $f(x) = \sqrt[3]{x} + \sec^2(x) + \pi^2$. Show all work.

(b) (10pts) Suppose the acceleration of an object at any time $t$ is given by $a(t) = 3t^2 - 4t$ m/s$^2$, $t \geq 0$. Find the velocity, $v(t)$, at any time $t$ if $v(1) = 1$ m/s. Show all work.

(c) (4pts) Which one of the choices below can be verified to be equivalent to $\int x \cos(x) \, dx$? (No justification necessary - Choose only one answer, copy down the entire answer in your bluebook.)

(A) $\frac{x^2}{2} \sin(x) + C$  
(B) $\cos(x) + x \sin(x) + C$  
(C) $x \sin(x) + C$  
(D) $\frac{\cos^2(x)}{2} + C$  
(E) $\sin(x) + C$

Solution: (a) (10pts) Note that

\[
\int (x^{1/3} + \sec^2(x) + \pi^2) \, dx = \frac{x^{4/3} + 1}{1/3 + 1} + \tan(x) + \pi^2 x + C = \frac{3}{4} x^{4/3} + \tan(x) + \pi^2 x + C
\]

(b) (10pts) Recall that $a(t) = v'(t)$ thus

\[
v(t) = \int a(t) \, dt = \int (3t^2 - 4t) \, dt = t^3 - 2 \cdot \frac{t^2}{2} + C = t^3 - 2t^2 + C
\]

and $v(1) = 1$ implies

\[
1 = v(1) = 1^3 - 2 \cdot 1^2 + C \Rightarrow 1 = -1 + C \Rightarrow C = 2 \Rightarrow v(t) = t^3 - 2t^2 + 2
\]
(c)(4pts) Choice B. Note that

$$
\frac{d}{dx} [\cos(x) + x \sin(x) + C] = -\sin(x) + \sin(x) + x \cos(x) + 0 = x \cos(x) \Rightarrow \int x \cos(x) \, dx = \cos(x) + x \sin(x) + C
$$