1. (24 points) Sketch a graph of $f(x)$ that satisfies all of the given conditions. Clearly label each aspect.

(a) Vertical asymptote at $x = 0$.
(b) $f'(x) > 0$ if $x < -2$.
(c) $f'(x) < 0$ if $x > -2$, $x \neq 0$.
(d) $f''(x) < 0$ if $x < 0$.
(e) $f''(x) > 0$ if $x > 0$.
(f) $f(-2)$ is an optimum.
(g) $\lim_{x \to -\infty} f(x) = 0$.
(h) $\lim_{x \to -\infty} [f(x) - x] = 0$.

Solution:
2. (24 points) Suppose you were provided with 1200 cm$^2$ of material. Your task is to build a square based box, with closed top, that can hold more than any other similar box built by the competition.

(a) This problem is seeking a maximum or a minimum what?

(b) Draw and label a picture for this situation.

(c) Create a single variable equation to be optimized.

(d) Find any proposed optimum locations from the equation in part (c).

(e) Verify the optimum location found in part (d) is the proper kind of optimum named in part (a).

(f) What is the value of the optimum sought?

Solution:

(a) Maximum Volume.

(b) 

(c) \[ V(x, h) = x^2 h, \quad 2x^2 + 4xh = 1200 \quad \Rightarrow \quad h = \frac{600 - x^2}{2x}. \quad V(x) = x^2 \cdot \frac{600 - x^2}{2x} \quad \text{or} \quad V(x) = 300x - \frac{1}{2}x^3 \]

(d) \[ V'(x) = 300 - \frac{3}{2}x^2. \quad V'(x) = 0 \quad \Rightarrow \quad x = 10\sqrt{2}. \]

(e) By the second-derivative test: \[ V''(x) = -3x, \quad \text{and} \quad V''(10\sqrt{2}) < 0. \] Therefore \( x = 10\sqrt{2} \) is the location of a maximum.

(f) \[ V(x, h) = x^2 h = 2000\sqrt{2}. \]
3. (15 points) Consider the function: \( f(x) = \frac{1}{2}x^4 - 4x \). It has only one critical number.

(a) Find the critical number for \( f(x) \).

(b) Use the second-derivative test to categorize the critical number as minimums or maximums.

(c) Use Newton’s Method with \( x_1 = 1 \) to find \( x_2 \) as a rational number estimate of the critical number.

Solution:

(a) \( y' = 2x^3 - 4 \). \( y' = 0 \) \( \implies x = \sqrt[3]{2} \).

(b) \( y''(x) = 6x^2 \). \( y''(\sqrt[3]{2}) > 0 \) \( \implies x = \sqrt[3]{2} \) is the location of a minimum.

(c) \( x_2 = 1 - \frac{2}{6} = \frac{4}{3} \)

4. (10 points) Suppose the acceleration of a particle is given by \( a(t) = 2\sin(t) - \sec^2(t) \) and velocity at time \( t = 0 \) is \(-6\). Find the velocity of the particle at time \( t \).

Solution: \( V''(t) = a(t) = 2\sin(t) - \sec^2(t) \implies V(t) = -2\cos(t) - \tan(t) + C \)

\( V(0) = -6 \implies C = -4 \) or \( V(t) = -2\cos(t) - \tan(t) - 4 \)

5. (11 points) (curve sketching) Name any and all asymptotes for \( y = \frac{x^3}{(x + 1)^2} \)

Solution: \( \lim_{x \to -1^+} \frac{x^3}{(x + 1)^2} = \frac{-1}{0^+} \to -\infty \implies x = -1 \) is a vertical asyptote.

\[
\begin{array}{c|c}
X^2 + 2X + 1 & X - 2 \\
\hline
-X^3 - 2X^2 - X & \hline
\end{array}
\]

This long division implies \( X - 2 \) is a slant asymptote.
6. (16 points) No work needs to be shown for this problem. For parts I - III consider the graph below:

I. If Newton’s Method is used with \( x_0 = 0 \) then what happens?

A. \( x_n \rightarrow 2 \).  
B. \( x_n \rightarrow 6 \).  
C. \( f'(x_0) = 0 \).  
D. Oscillation between values.  
E. \( x_n \rightarrow -\infty \).  
F. None of the above.

II. If Newton’s Method is used with \( x_0 = 5 \) then what happens?

A. \( x_n \rightarrow 2 \).  
B. \( x_n \rightarrow 6 \).  
C. \( f'(x_0) = 0 \).  
D. Oscillation between values.  
E. \( x_n \rightarrow -\infty \).  
F. None of the above.

III. If Newton’s Method is used with \( x_0 = 6 \) then what happens?

A. \( x_n \rightarrow 2 \).  
B. \( x_n \rightarrow 6 \).  
C. \( f'(x_0) = 0 \).  
D. Oscillation between values.  
E. \( x_n \rightarrow -\infty \).  
F. None of the above.

IV. Suppose \( f''(x) \) is continuous on \( (-\infty, \infty) \). If \( f'(2) = 0 \) and \( f''(2) = 0 \), then what can be said about \( f(x) \)? Choose all that apply. This question is NOT referring to the graph above.

A. \( f(x) \) has a maximum at \( x = 2 \).  
B. \( f(x) \) has a point of inflection at \( x = 2 \).  
C. \( f(x) \) has a horizontal tangent at \( x = 2 \).  
D. \( f(x) \) has a local minimum at \( x = 2 \).  
E. \( f(x) \) is undefined at \( x = 2 \).  
F. None of the above.

Solution:

I. E  
II. B  
III. B  
IV. C