1. [35 pts] Consider the function \( f(x) = x^4 - 6x^2 \).

   (a) [2 pts] What is the domain of \( f(x) \)?
   (b) [3 pts] Is \( f(x) \) odd, even or neither? Justify your answer algebraically.
   (c) [3 pts] Find all the \( x \)-intercepts and \( y \)-intercepts of \( f(x) \), if any.
   (d) [2 pts] Find all the asymptotes of \( f(x) \), if any.
   (e) [2 pts] Find \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).
   (f) [2 pts] Find \( f'(x) \). Check your answer very carefully before proceeding.
   (g) [2 pts] Find \( f''(x) \). Check your answer very carefully before proceeding.
   (h) [3 pts] Where is \( f(x) \) increasing and where is \( f(x) \) decreasing? Write your answer using interval notation.
   (i) [3 pts] Find all local extrema of \( f(x) \), if it possesses any.
   (j) [3 pts] Where is \( f(x) \) concave up and where is \( f(x) \) concave down? Write your answer using interval notation.
   (k) [3 pts] Find all inflection points of \( f(x) \), if it possesses any.
   (l) [7 pts] Sketch the graph of \( f(x) \). Label all intercepts, relative extrema (if any) and inflection points (if any).

**Solution:**

(a) \( f(x) \) is a polynomial so its domain is all real numbers \( \mathbb{R} \) or \( (-\infty, \infty) \)

(b) \( f(-x) = (-x)^4 - 6(-x)^2 = x^4 - 6x^2 = f(x) \implies f(x) \) is even (symmetric with respect to \( y \)-axis)

(c) \( f(0) = 0 \) so \( y \)-intercept is \((0, 0)\). \( x^4 - 6x^2 = 0 \implies x^2(x^2 - 6) = x^2(x + \sqrt{6})(x - \sqrt{6}) = 0 \implies x = 0, \pm \sqrt{6} \)

   so the \( x \)-intercepts are \((0, 0), (-\sqrt{6}, 0) \) and \((\sqrt{6}, 0)\).

(d) There are no asymptotes since \( f(x) \) is a polynomial.

(e) \( \lim_{x \to \infty} x^4 \left(1 - \frac{6}{x^2}\right) = \lim_{x \to -\infty} x^4 \left(1 - \frac{6}{x^2}\right) = \infty \)

(f) \( f'(x) = 4x^3 - 12x \)

(g) \( f''(x) = 12x^2 - 12 \)

(h) \( f'(x) = 4x(x^2 - 3) = 4x(x + \sqrt{3})(x - \sqrt{3}) \) so the critical points are \( x = 0, \pm \sqrt{3} \) and the sign of \( f' \) is shown below:

\[
\begin{array}{c|ccc}
 x & - & + & + \\
\hline 
-\sqrt{3} & 0 & \sqrt{3} \\
\end{array}
\]

From the chart, \( f(x) \) is increasing on \((-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) \) and decreasing on \((-\infty, -\sqrt{3}) \cup (0, \sqrt{3}) \)

(i) \( f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 = 9 - 18 = -9 \) and \( f(\sqrt{3}) = (\sqrt{3})^4 - 6(\sqrt{3})^2 = 9 - 18 = -9 \) are local minima and \( f(0) = 0 \) is a local maximum.

(j) \( f''(x) = 12(x^2 - 1) = 12(x + 1)(x - 1) \) and the sign of \( f'' \) is shown below:

\[
\begin{array}{c|ccc}
 x & + & - & + \\
\hline 
-1 & 0 & 1 \\
\end{array}
\]

From the chart, \( f(x) \) is concave up on \((-\infty, -1) \cup (1, \infty) \) and concave down on \((-1, 1) \).

(k) Inflection points are \((-1, f(-1)) = (-1, -5) \) and \((1, f(1)) = (1, -5) \).
2. [20 pts] Consider the function \( p(x) = x^3 - 3x + 4 \).

(a) Set up the equation that uses Newton’s method to find the root(s) of the function.

(b) Using your answer to part (a), if \( x_1 = 0 \), find \( x_2 \).

(c) Suppose \( x_1 \) is chosen such that \( x_2 = -1 \). What is \( x_3 \) in this case? Explain briefly.

(d) Set up the equation that uses Newton’s method to solve the equation \( p(x) = 3 \).

**Solution:**

(a) \( p'(x) = 3x^2 - 3 \) so Newton’s method is 
\[
  x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = x_n - \frac{x_n^3 - 3x_n + 4}{3x_n^2 - 3}.
\]

(b) 
\[
  x_2 = 0 - \frac{0^3 - 3(0) + 4}{3(0^2) - 3} = \frac{4}{3}.
\]

(c) \( x_3 \) cannot be computed as the denominator in the equation for Newton’s method would be 0.

(d) The solutions to \( p(x) = 3 \) are the same as the roots of \( f(x) = x^3 - 3x + 1 = 0 \) since \( x^3 - 3x + 4 = 3 \iff x^3 - 3x + 1 = 0 \). With \( f(x) = x^3 - 3x + 1 \), \( f'(x) = 3x^2 - 3 \) and Newton’s method is 
\[
  x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}.
\]

3. [15 pts] An open (no lid) rectangular box is to be constructed with a square base and a volume of 32 cm\(^3\). Find the dimensions of the box that require the least amount of material.

**Solution:**

Let the length of the base of the box be \( x \) and its height be \( y \). Then its volume is \( V = x^2y \). We want to use the least amount of material to build the box, meaning that we want the box to have the smallest surface area, \( S \), with \( S = x^2 + 4xy \). We need to eliminate one of the variables from this equation, accomplished by noting that \( x^2y = 32 \iff y = \frac{32}{x^2} \). Thus 
\[
  S(x) = x^2 + 4x \left( \frac{32}{x^2} \right) = x^2 + \frac{128}{x}.
\]
This is the function we need to minimize on the domain \( x > 0 \). Setting \( S'(x) = 2x - \frac{128}{x^2} \) to 0 yields 
\[
  2x - \frac{128}{x^2} = 0 \iff x - \frac{64}{x} = 0 \iff x^2 - 64 = 0 \iff x = \pm 4.
\]
This is satisfied if \( x^2 - 64 = 0 \iff x = 4 \). Since \( S''(4) = 2 + \frac{128}{4^2} > 0 \), \( x = 4 \) does yield a minimum. Furthermore, with \( x = 4 \), \( y = \frac{32}{4^2} = 2 \) so the dimensions are 
\[
  4 \times 4 \times 2 \text{ cm}.
\]

4. [20 pts] The following problems are unrelated.
(a) [6 pts] Use the second derivative test to classify the relative extrema of the function \( g(x) = \frac{x}{x^2 + 1} \). (Hint: you don’t necessarily have to simplify \( g'' \))

(b) [6 pts] Find the most general antiderivative of \( f(s) = \frac{5}{\sqrt{s^2}} + \frac{2}{\sqrt{s^3}} \).

(c) [8 pts] A particle is moving with an acceleration \( a(t) = 10 \sin t + 3 \cos t \). When time \( t = 0 \), the particle’s position is 0 and when \( t = 2\pi \) its position is 12. Find its position when \( t = \frac{5\pi}{2} \).

**SOLUTION:**

(a) \[
g'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(1 - x)(1 + x)}{(x^2 + 1)^2}
\]
\[
g''(x) = \frac{(x^2 + 1)^2(-2x) - (1 - x^2)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}
\]

Setting \( g' \) equal to zero, the critical points are seen to be \( x = \pm 1 \). Then

\[
g''(1) = \frac{2(1)(2 - 3)}{(1^2 + 1)^3} = -\frac{1}{2} < 0 \implies g(1) = \frac{1}{2} \text{ is a relative maximum}
\]
\[
g''(-1) = \frac{2(-1)((-1)^2 - 3)}{((-1)^2 + 1)^3} = \frac{1}{2} > 0 \implies g(-1) = -\frac{1}{2} \text{ is a relative minimum}
\]

(b) \( f(s) = \frac{5}{\sqrt{s^2}} + \frac{2}{\sqrt{s^3}} = 5s^{-2/3} + 2s^{-3/2} \) so that

\[
F(s) = 5 \frac{s^{-2/3} + 1}{(-2/3 + 1)} + 2 \frac{s^{-3/2} + 1}{(-3/2 + 1)} + C = 15s^{1/3} - 4s^{-1/2} + C = 15\sqrt{s} - \frac{4}{\sqrt{s}} + C
\]

(c) \[
v(t) = \int (10 \sin t + 3 \cos t) \, dt = -10 \cos t + 3 \sin t + C_1
\]
\[
\implies s(t) = \int (-10 \cos t + 3 \sin t + C_1) \, dt = -10 \sin t - 3 \cos t + C_1 t + C_2
\]
\[
s(0) = -10 \sin 0 - 3 \cos 0 + C_1(0) + C_2 = 0 \implies -3 + C_2 = 0 \implies C_2 = 3
\]
\[
s(2\pi) = -10 \sin 2\pi - 3 \cos 2\pi + C_1(2\pi) + 3 = 12 \implies C_1 = \frac{6}{\pi}
\]

Thus \( s(t) = -10 \sin t - 3 \cos t + \frac{6}{\pi} t + 3 \) and \( s \left( \frac{5\pi}{2} \right) = -10 \sin \frac{5\pi}{2} - 3 \cos \frac{5\pi}{2} + \frac{6}{\pi} \frac{5\pi}{2} + 3 = -10 + 15 + 3 = 8 \)

5. [10 pts] In your bluebook, write **TRUE** if the statement is true and write **FALSE** is the statement is false. Do not abbreviate with T or F. No justification required and no partial credit given.

(a) If \( f \) is increasing for all \( x < c \) and \( f \) is decreasing for all \( x > c \), then \( c \) must be a local maximum.

(b) The concavity of a function can only change at inflection points.

(c) \( y = \frac{1}{2}x - 5 \) is a slant asymptote of the graph of \( f(x) = \frac{x^2 - 10x + 8}{2x} \).

(d) Suppose \( f(x) \) exists at the point \( c \) but \( f'(x) \) does not exist at the point \( c \). Then the graph of \( f \) must possess a cusp at \((c, f(c))\).
(e) The graph of \( f(x) = \frac{x^2 + x - 6}{x^2 - 9} \) possesses a vertical asymptote at \( x = -3 \).

**SOLUTION:**

(a) **FALSE** \( f(c) \) (y-value) would be a local maximum, not \( c \) itself; or \( c \) may not be in the domain of \( f \), that is, \( f(c) \) need not exist.

(b) **FALSE** Concavity can change across an asymptote

(c) **TRUE** \( f(x) = \frac{1}{2}x - 5 + \frac{4}{x} \)

(d) **FALSE** \( f \) could have a vertical tangent at \( c \)

(e) **FALSE** \( f \) has a removable discontinuity at \(-3\) since \( \lim_{x \to -3} f(x) \neq \pm \infty \)