1. (19 pts) Parts (a) and (b) are not related.

   (a) For $f(x) = \sqrt{x^2 - 4}$ and $g(x) = \dfrac{1}{x}$, identify the composite function $(f \circ g)(x)$ and its domain. Express the domain in interval form.

Solution:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \boxed{\sqrt{\frac{1}{x^2} - 4}}$$

The domain of $g(x)$ is $x \neq 0$.

The domain of $\sqrt{\dfrac{1}{x^2} - 4}$ is the set of all x values such that $\dfrac{1}{x^2} - 4 \geq 0$.

$$0 \leq \frac{1}{x^2} - 4$$

$$4 \leq \frac{1}{x^2}$$

$$x^2 \leq \frac{1}{4}$$

$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$

Therefore, the domain of $(f \circ g)(x)$ is $\boxed{[-1/2, 0) \cup (0, 1/2]}$

(b) The graph of $y = \cos x$ is transformed in the following three steps, in the specified order:

i) Stretched horizontally by a factor of 2
ii) Shifted horizontally by 3 units to the right
iii) Reflected across the x-axis

After each of the three transformations, what is the equation of the resulting graph? Note that no actual graphing is required in this problem.

i. Equation of the graph after transformation (i):

Solution:

A horizontal stretch by a factor of two is achieved by replacing x with $x/2$.

$$\boxed{y = \cos\left(\frac{1}{2}x\right)}$$

ii. Equation of the graph after transformations (i) and (ii):

Solution:

A horizontal shift by 3 units to the right is achieved by replacing x with $(x - 3)$.

$$\boxed{y = \cos\left(\frac{1}{2}(x-3)\right)}$$

iii. Equation of the graph after transformations (i), (ii), and (iii):

Solution:

Reflection about the x-axis is achieved by multiplying the function by -1.

$$\boxed{y = -\cos\left(\frac{1}{2}(x-3)\right)}$$

2. (31 pts) Evaluate the following limits. If you use a named theorem, state the name as part of your solution.

(a) $\lim_{x \to 0} \frac{x\cot(3x)}{x-4}$

Solution:

$$\lim_{x \to 0} \frac{x\cot(3x)}{x-4} = \lim_{x \to 0} \frac{x}{x-4} \cdot \frac{\cos(3x)}{\sin(3x)}$$

$$= \lim_{x \to 0} \frac{3}{3} \cdot \frac{x}{x-4} \cdot \frac{\cos(3x)}{\sin(3x)}$$

$$= \lim_{x \to 0} \frac{\cos(3x)}{3(x-4)} \cdot \frac{3x}{\sin(3x)}$$

$$= \lim_{x \to 0} \frac{\cos(3x)}{3(x-4)} \cdot \lim_{x \to 0} \frac{3x}{\sin(3x)}$$

$$= \frac{\cos(0)}{3(0-4)} \cdot (1) = \boxed{-\frac{1}{12}}$$

(b) $\lim_{x \to -4} \frac{x+4}{\sqrt{x^2+9}-5}$

Solution:

$$\lim_{x \to -4} \frac{x+4}{\sqrt{x^2+9}-5} = \lim_{x \to -4} \frac{x+4}{\sqrt{x^2+9}-5} \cdot \frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}$$

$$= \lim_{x \to -4} \frac{(x+4)\left(\sqrt{x^2+9}+5\right)}{(x^2+9)-25}$$

$$= \lim_{x \to -4} \frac{(x+4)\left(\sqrt{x^2+9}+5\right)}{(x-4)(x+4)}$$

$$= \lim_{x \to -4} \frac{\sqrt{x^2+9}+5}{x-4}$$

$$= \frac{\sqrt{(-4)^2+9}+5}{-4-4} = \frac{10}{-8} = \boxed{-\frac{5}{4}}$$

(c) $\lim_{x \to 2} (x-2)^2 \sin\left(\frac{1}{x^2 - x - 2}\right)$

Solution:

$$-1 \leq \sin\left(\frac{1}{x^2 - x - 2}\right) \leq 1$$

$$-(x-2)^2 \leq (x-2)^2 \sin\left(\frac{1}{x^2 - x - 2}\right) \leq (x-2)^2$$

Note that the quantity $(x-2)^2$ is nonnegative, so that the direction of the inequalities did not change.

$\lim_{x \to 2} [-(x-2)^2] = \lim_{x \to 2} (x-2)^2 = 0$

Therefore, the **Squeeze Theorem** indicates that $\lim_{x \to 2} (x-2)^2 \sin\left(\frac{1}{x^2 - x - 2}\right) = \boxed{0}$

3. (32 pts) Consider the function $h(x) = \dfrac{(x-1)\sqrt{4x^4+1}}{x^3 - 3x^2 + 2x}$.

(a) Identify all values of x, if any, for which $y = h(x)$ has a removable discontinuity. Support your answer by evaluating the appropriate limit(s).

Solution:

$$h(x) = \frac{(x-1)\sqrt{4x^4+1}}{x^3 - 3x^2 + 2x} = \frac{(x-1)\sqrt{4x^4+1}}{x(x^2 - 3x + 2)} = \frac{(x-1)\sqrt{4x^4+1}}{x(x-1)(x-2)}$$

Therefore,

$$h(x) = \frac{\sqrt{4x^4+1}}{x(x-2)} \text{ for } x \neq 0, 1, 2$$

(The preceding simplified expression for $h(x)$ will be useful in parts (b) and (c) as well.)

Since the simplified expression for $h(x)$ does not produce division by zero for $x = 1$, a two-sided limit can be evaluated.

$$\lim_{x \to 1} \frac{\sqrt{4x^4+1}}{x(x-2)} = \frac{\sqrt{(4)(1)+1}}{(1)(1-2)} = \frac{\sqrt{5}}{-1} = -\sqrt{5}$$

Since $h(x)$ approaches a finite two-sided limit as x approaches 1, h has a removable discontinuity at $\boxed{x = 1}$

(b) Find the equation of each vertical asymptote of $y = h(x)$, if any exist. Support your answer by evaluating the appropriate limit(s).

Solution:

From part (a), we know that $h(x) = \dfrac{\sqrt{4x^4+1}}{x(x-2)} \quad , \quad x \neq 0, 1, 2.$

Since this simplified function expression produces division by zero for $x = 0$ and $x = 2$, one-sided limits must be evaluated.

$$\lim_{x\to 0^-} \frac{\sqrt{4x^4+1}}{x(x-2)} \to \frac{\sqrt{(4)(0)+1}}{(0^-)(0-2)} \to \frac{1}{(0^-)(-2)} \to \infty$$

$$\lim_{x\to 0^+} \frac{\sqrt{4x^4+1}}{x(x-2)} \to \frac{\sqrt{(4)(0)+1}}{(0^+)(0-2)} \to \frac{1}{(0^+)(-2)} \to -\infty$$

Either of the two preceding limits is sufficient to establish that h has a vertical asymptote at $\boxed{x = 0}$

$$\lim_{x\to 2^-} \frac{\sqrt{4x^4+1}}{x(x-2)} \to \frac{\sqrt{(4)(2^4)+1}}{(2)(0^-)} \to \frac{\sqrt{65}}{(2)(0^-)} \to -\infty$$

$$\lim_{x\to 2^+} \frac{\sqrt{4x^4+1}}{x(x-2)} \to \frac{\sqrt{(4)(2^4)+1}}{(2)(0^+)} \to \frac{\sqrt{65}}{(2)(0^+)} \to \infty$$

Either of the two preceding limits is sufficient to establish that h has a vertical asymptote at $\boxed{x = 2}$

(Note that there is a vertical asymptote at a particular value of x if **at least one** of the corresponding one-sided limits is infinite.)

(c) Find the equation of each horizontal asymptote of $y = h(x)$, if any exist. Support your answer by evaluating the appropriate limit(s). *(Reminder: You may not use L'Hôpital's Rule or dominance of powers arguments to evaluate limits on this exam.)*

Solution:

From part (a), we know that $h(x) = \dfrac{\sqrt{4x^4+1}}{x(x-2)}$, $x \neq 0, 1, 2.$

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{\sqrt{4x^4+1}}{x(x-2)} &= \lim_{x\to\pm\infty} \frac{\sqrt{x^4(4+1/x^4)}}{x^2-2x} \\
&= \lim_{x\to\pm\infty} \frac{\sqrt{x^4}\,\sqrt{4+1/x^4}}{x^2-2x} \\
&= \lim_{x\to\pm\infty} \frac{x^2\,\sqrt{4+1/x^4}}{x^2-2x} \\
&= \lim_{x\to\pm\infty} \frac{x^2\,\sqrt{4+1/x^4}}{x^2-2x} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to\pm\infty} \frac{\sqrt{4+1/x^4}}{1-2/x} \\
&= \frac{\sqrt{4+0}}{1-0} = 2
\end{aligned}$$

Therefore, $h(x)$ has one horizontal asymptote: $\boxed{y = 2}$

4. (18 pts) Parts (a) and (b) are not related.

(a) For what pair of values a and b is the following function $u(x)$ continuous at $x = 3$? Support your answer using the definition of continuity, which includes evaluating the appropriate limits.

$$u(x) = \begin{cases} x + \dfrac{a}{x} & , \quad x < 3 \\ 2b + 2 & , \quad x = 3 \\ x + b & , \quad x > 3 \end{cases}$$

Solution:

By definition, in order for $u(x)$ to be continuous at $x = 3$, the following must be true:

$$\lim_{x \to 3^-} u(x) = \lim_{x \to 3^+} u(x) = u(3)$$

$$\lim_{x \to 3^-} u(x) = \lim_{x \to 3^-} \left(x + \frac{a}{x}\right) = 3 + \frac{a}{3}$$

$$\lim_{x \to 3^+} u(x) = \lim_{x \to 3^+} (x + b) = 3 + b$$

$$u(3) = 2b + 2$$

Therefore, we must have: $3 + \dfrac{a}{3} = 3 + b = 2b + 2.$

$$3 + b = 2b + 2 \quad \Rightarrow \quad \boxed{b = 1}$$

$$3 + \frac{a}{3} = 3 + b = 3 + 1 = 4 \quad \Rightarrow \quad \frac{a}{3} = 1 \quad \Rightarrow \quad \boxed{a = 3}$$

(b) Use a Calculus 1 theorem to establish that the equation $v(x) = (x-1)(x+2) + \sin^2 x = 0$ has at least one solution on the interval $(0, \pi/2)$. Name the theorem that is used and verify that the conditions for applying it to this problem are satisfied.

Solution:

$$v(0) = (0-1)(0+2) + 0^2 = -2 < 0$$

$$v(\pi/2) = (\pi/2 - 1)(\pi/2 + 2) + 1^2 > 0$$

(Note that $\pi/2 \approx 3.14/2 = 1.57 > 1$)

Therefore, since $v(x)$ is continuous on $[0, \pi/2]$, $v(0) < 0$, and $v(\pi/2) > 0$, the **Intermediate Value Theorem** indicates that $v(x) = 0$ has at least one solution on the interval $(0, \pi/2)$.