1. (32 pts) The position function of a particle is given by $s(t) = 4\sqrt{t} - t$ on the interval $1 \leq t \leq 16$, where position is in meters and time is in seconds.

   (a) Determine the particle's velocity function $v(t)$. Include the correct unit of measurement.

   Solution:

   $s(t) = 4\sqrt{t} - t = 4t^{1/2} - t$

   $v(t) = s'(t) = 2t^{-1/2} - 1 = \boxed{\frac{2}{\sqrt{t}} - 1 \text{ m/s}}, \quad 1 < t < 16$

   (b) Determine the total distance traveled by the particle on the interval $1 \leq t \leq 16$. Include the correct unit of measurement.

   Solution:

   $v(t) = \frac{2}{\sqrt{t}} - 1 = 0 \quad \Rightarrow \quad \frac{2}{\sqrt{t}} = 1 \quad \Rightarrow \quad \sqrt{t} = 2 \quad \Rightarrow \quad t = 4$

   Since $v(t)$ changes sign at $t = 4$, we need to calculate the distances traveled during the time intervals $[1, 4]$ and $[4, 16]$ separately, and add those results together.

   Distance traveled on time interval $[1, 4] = |s(4) - s(1)|$

   Distance traveled on time interval $[4, 16] = |s(16) - s(4)|$

   $s(1) = 4\sqrt{1} - 1 = 4 - 1 = 3$

   $s(4) = 4\sqrt{4} - 4 = 8 - 4 = 4$

   $s(16) = 4\sqrt{16} - 16 = 16 - 16 = 0$

   Distance traveled on time interval $[1, 4] = |s(4) - s(1)| = |4 - 4| = 1$

   Distance traveled on time interval $[4, 16] = |s(16) - s(4)| = |0 - 4| = 4$

   Therefore, the total distance traveled between $t = 1$ and $t = 16$ seconds is $1 + 4 = \boxed{5 \text{ m}}$

(c) i. Verify that all hypotheses of the Mean Value Theorem are satisfied for the given position function $s(t) = 4\sqrt{t} - t$ on the interval $1 \le t \le 16$.

Solution:

$s(t)$ is $\boxed{\text{continuous on } [1, 16]}$ and $\boxed{\text{differentiable on } (1, 16)}$

ii. Use the Mean Value Theorem to determine all time values c on the interval $1 \le t \le 16$, if any, for which the instantaneous velocity of the particle equals the average velocity of the particle on that interval. Include the correct unit of measurement.

Solution:

The Mean Value Theorem states that since the hypotheses have been satisfied, there exists at least one number c on the interval $(1, 16)$ such that

$$s'(c) = \frac{s(16) - s(1)}{16 - 1} = \frac{0 - 3}{15} = -\frac{1}{5}$$

Therefore, using the result from part (a), we have

$$v(c) = \frac{2}{\sqrt{c}} - 1 = -\frac{1}{5}$$

$$\frac{2}{\sqrt{c}} = \frac{4}{5}$$

$$\sqrt{c} = \frac{5}{2}$$

$$\boxed{c = \frac{25}{4} \text{ sec}}$$

2. (11 pts) Let v represent a person's walking speed, expressed in miles per hour, and let p represent the corresponding walking pace, expressed in minutes per mile. The pace can be expressed as the following function of speed:

$$p(v) = \frac{60}{v}, \quad v > 0$$

(a) Find the linearization $L(v)$ that approximates $p(v)$ near $v = 4$.

Solution:

$$p(v) \approx L(v) = p(4) + p'(4)(v - 4)$$

$$p(4) = \frac{60}{4} = 15$$

$$p'(v) = -\frac{60}{v^2} \quad \Rightarrow \quad p'(4) = -\frac{60}{4^2} = -\frac{15}{4}$$

$$L(v) = \boxed{15 - \frac{15}{4}(v - 4)}$$

(b) Use your linearization from part (a) to estimate the walking pace of a person moving at 4.2 miles per hour. Include the correct unit of measurement. You must use linearization to earn credit.

Solution:

$$p(4.2) \approx L(4.2) = 15 - \frac{15}{4}(4.2 - 4) = 15 - \frac{15}{4} \cdot \frac{1}{5} = 15 - \frac{3}{4} = \boxed{\text{14.25 minutes per mile}}$$

3. (26 pts) Consider the function $f(x) = \sin x + \cos^2 x$ on the interval $[0, 2\pi/3]$.

   (a) Identify all critical numbers of f on the specified interval.

   Solution:

   Critical numbers are values of x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist. There are no critical numbers of the latter type for this function.

   $f'(x) = \cos x + 2\cos x \cdot (-\sin x) = \cos x(1 - 2\sin x) = 0$

   $x = \pi/2$ is the only value of x in the specified domain that satisfies $\cos x = 0$.

   $x = \pi/6$ is the only value of x in the specified domain that satisfies $\sin x = 1/2$.

   Therefore, the critical numbers of f on the specified domain are $\boxed{\frac{\pi}{6}, \frac{\pi}{2}}$

   (b) Use the Closed Interval Method to find the absolute maximum and minimum values of f on the specified interval. Clearly identify all x values that are associated with the absolute maximum function value and all x values that are associated with the absolute minimum function value. Note that $\sqrt{3} \approx 1.7$.

   Solution:

   The Closed Interval Method involves evaluating the function at the domain endpoints and at the critical numbers.

   $f(0) = 1 + 0^2 = 1$

   $f(\pi/6) = 1/2 + (\sqrt{3}/2)^2 = 5/4$

   $f(\pi/2) = 1 + 0^2 = 1$

   $f(2\pi/3) = \sqrt{3}/2 + (-1/2)^2 = \sqrt{3}/2 + 1/4 = (2\sqrt{3} + 1)/4 \approx 4.4/4$

   Since $1 < (2\sqrt{3} + 1)/4 < 5/4$, the absolute maximum value of f occurs at the point $\boxed{(\pi/6,\, 5/4)}$ and the absolute minimum value of f occurs at the points $\boxed{(0, 1)}$ and $\boxed{(\pi/2,\, 1)}$

4. (26 pts) Find an equation of each tangent line, if any, to the curve $2x^3 + 2y^2 = 5xy$ at $x = 1$.

Solution:

We begin by executing implicit differentiation, as follows:

$$\frac{d}{dx}[2x^3 + 2y^2] = \frac{d}{dx}[5xy]$$

$$6x^2 + 4yy' = 5(xy' + y)$$

$$(4y - 5x)y' = 5y - 6x^2$$

$$y' = \frac{5y - 6x^2}{4y - 5x}$$

Next, we determine the y-coordinate value of each point on the given curve that has an x-coordinate value of 1.

$$(2)(1)^3 + 2y^2 = (5)(1)y$$

$$2y^2 - 5y + 2 = 0$$

The quadratic formula indicates that $y = \dfrac{-(-5) \pm \sqrt{(-5)^2 - (4)(2)(2)}}{(2)(2)} = \dfrac{5 \pm 3}{4} \quad \Rightarrow \quad y = \dfrac{1}{2}, 2$

So, there are two points on the given curve at $x = 1$: $(1,\ 1/2)$ and $(1, 2)$. The slopes of the tangent lines at those points are:

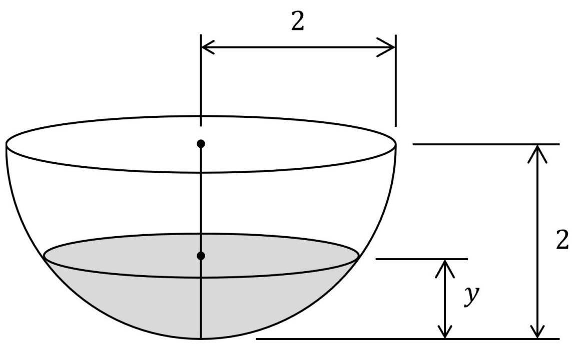
$$x = 1,\ y = 1/2: \quad y' = \frac{(5)(1/2) - (6)(1)^2}{(4)(1/2) - (5)(1)} = \frac{5/2 - 6}{-3} = \frac{5 - 12}{-6} = \frac{7}{6}$$

$$x = 1,\ y = 2: \quad y' = \frac{(5)(2) - (6)(1)^2}{(4)(2) - (5)(1)} = \frac{10 - 6}{3} = \frac{4}{3}$$

Therefore, point-slope forms of the tangent lines are $\boxed{y - \dfrac{1}{2} = \dfrac{7}{6}(x - 1)}$ and $\boxed{y - 2 = \dfrac{4}{3}(x - 1)}$

5. (18 pts) Two bowls of identical size are both hemispheres of radius 2 ft. When such a bowl contains water having a depth of y ft, as depicted below, the corresponding volume of water in the bowl is given by the following function:

$V = \pi y^2(2 - y/3)\ , \quad 0 \le y \le 2$



(a) Suppose bowl A is being filled at a constant rate of $8\pi/9$ cubic ft per minute. How fast is the depth of the water increasing when the water is 1 ft deep? Simplify your answer fully and include the correct unit of measurement.

(b) Suppose bowl B is being filled in such a way that its water depth is increasing at a constant rate of $1/3$ ft per minute. How fast is the volume of the water increasing when the water is 1 ft deep? Simplify your answer fully and include the correct unit of measurement.

Solution:

The Chain Rule indicates that $\dfrac{dV}{dt} = \dfrac{dV}{dy} \cdot \dfrac{dy}{dt}$

$$V = \pi y^2 \left(2 - \frac{y}{3}\right) = 2\pi y^2 - \frac{\pi}{3}\, y^3$$

$$\frac{dV}{dy} = 4\pi y - \pi y^2 \quad \Rightarrow \quad \left.\frac{dV}{dy}\right|_{y=1} = 4\pi - \pi = 3\pi$$

(a) $\dfrac{dV}{dt} = \dfrac{8\pi}{9}$

$$y = 1: \quad \frac{8\pi}{9} = 3\pi \cdot \frac{dy}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{8\pi}{9} \cdot \frac{1}{3\pi} = \boxed{\frac{8}{27} \text{ ft per min}}$$

(b) $\dfrac{dy}{dt} = \dfrac{1}{3}$

$$y = 1: \quad \frac{dV}{dt} = 3\pi \cdot \frac{1}{3} = \boxed{\pi \text{ cubic ft per min}}$$

6. (15 pts) Determine $g'(x)$ for the function $g(x) = \sqrt{3x+1}$ by using the definition of derivative. You must obtain g' by evaluating an appropriate limit to earn credit.

Solution:

The definition of derivative indicates that $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}$

Begin by multiplying the numerator and the denominator by the numerator's conjugate.

$$g'(x) = \lim_{h\to 0} \frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1}+\sqrt{3x+1}}{\sqrt{3(x+h)+1}+\sqrt{3x+1}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{3(x+h)+1})^2-(\sqrt{3x+1})^2}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})}$$

$$= \lim_{h\to 0} \frac{(3x+3h+1)-(3x+1)}{h(\sqrt{3x+3h+1}+\sqrt{3x+1})}$$

$$= \lim_{h\to 0} \frac{3h}{h(\sqrt{3x+3h+1}+\sqrt{3x+1})}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$= \frac{3}{\sqrt{3x+0+1}+\sqrt{3x+1}} = \boxed{\frac{3}{2\sqrt{3x+1}}}$$

7. (22 pts) Consider the function $h(x) = \dfrac{\sin x}{x(x - \pi/2)}$.

(a) Find the (x, y) coordinates of every removable discontinuity of $h(x)$, if any exist. Support your answer by evaluating the appropriate limit(s).

Solution:

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{\sin x}{x(x - \pi/2)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{(x - \pi/2)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{(x - \pi/2)} = (1)\left(\frac{1}{0 - \pi/2}\right) = -\frac{2}{\pi}$$

Since the preceding limit is finite, h has a removable discontinuity at the point $\boxed{\left(0, -\frac{2}{\pi}\right)}$

(b) Find the equation of every vertical asymptote of $y = h(x)$, if any exist. Support your answer by evaluating the appropriate limit(s).

Solution:

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{x(x - \pi/2)} \to \frac{\sin(\pi/2)}{(\pi/2)(0^-)} \to \frac{1}{(\pi/2)(0^-)} \to \frac{(2/\pi)}{0^-} \to -\infty$$

$$\lim_{x \to \frac{\pi}{2}^+} \frac{\sin x}{x(x - \pi/2)} \to \frac{\sin(\pi/2)}{(\pi/2)(0^+)} \to \frac{1}{(\pi/2)(0^+)} \to \frac{(2/\pi)}{0^+} \to \infty$$

Since at least one of the preceding two limits is infinite (in fact, both are infinite for this function), $f(x)$ has a vertical asymptote at $\boxed{x = \pi/2}$