1. (30 pts) Determine $\dfrac{dy}{dx}$ for each of the following.

   (a) $y = \sin^4(x^3)$

   (b) $x^2 + xy + y^3 = 4$

   (c) $y = \dfrac{2x^2+1}{x\cos x}$ After fully differentiating, do not algebraically simplify your answer any further.

Solution:

(a)

$$\frac{d}{dx}\left[\sin^4(x^3)\right] = 4\sin^3(x^3)\frac{d}{dx}\left[\sin(x^3)\right] = 4\sin^3(x^3)\cos(x^3)\frac{d}{dx}\left[x^3\right]$$

$$= 4\sin^3(x^3)\cos(x^3)(3x^2) = \boxed{12x^2\sin^3(x^3)\cos(x^3)}$$

(b)

$$\frac{d}{dx}\left[x^2+xy+y^3\right] = \frac{d}{dx}[4]$$

$$2x + xy' + y + 3y^2y' = 0$$

$$y'(x+3y^2) = -(2x+y)$$

$$y' = \boxed{-\frac{2x+y}{x+3y^2}}$$

(c)

$$\frac{d}{dx}\left[\frac{2x^2+1}{x\cos x}\right] = \frac{x\cos x\cdot\frac{d}{dx}[2x^2+1] - (2x^2+1)\cdot\frac{d}{dx}[x\cos x]}{(x\cos x)^2}$$

$$= \frac{x\cos x\cdot(4x) - (2x^2+1)\left(x\cdot\frac{d}{dx}[\cos x] + \cos x\cdot\frac{d}{dx}[x]\right)}{(x\cos x)^2}$$

$$= \boxed{\frac{x\cos x\cdot(4x) - (2x^2+1)(-x\sin x+\cos x)}{(x\cos x)^2}}$$

2. (25 pts) Parts (a) and (b) are unrelated.

   (a) The position function of Particle P is given by $s(t) = 2/t + t/2$, where s is in meters, t is in seconds, and $t \geq 1$.

      i. Find the particle's velocity function $v(t)$. Include the correct unit of measurement.

      ii. Find the distance traveled by the particle between $t = 1$ and $t = 8$ seconds. Include the correct unit of measurement.

   (b) The velocity function of Particle Q is given by $v(t) = \tan t - t^2/4$, where v is in miles per hour, t is in hours, and $3\pi/2 < t < 5\pi/2$.

      i. Find the particle's acceleration function $a(t)$. Include the correct unit of measurement.

      ii. Find the acceleration of the particle at $t = 2\pi$ hours. Include the correct unit of measurement.

Solution:

(a) i. $s(t) = 2t^{-1} + \frac{1}{2}t$

$$v(t) = s'(t) = -2t^{-2} + \frac{1}{2} = \boxed{-\frac{2}{t^2} + \frac{1}{2} \text{ m/s}}$$

ii. $s'(t) = 0 = -\frac{2}{t^2} + \frac{1}{2}$

$t^2 = 4 \quad \Rightarrow \quad t = 2 \qquad$ ($t = -2$ is not in the stated domain of s)

The sign of $s'(t)$ changes at $t = 2$. Therefore,

$$D = |s(2) - s(1)| + |s(8) - s(2)| = \left|(1+1) - \left(2 + \frac{1}{2}\right)\right| + \left|\left(\frac{1}{4} + 4\right) - (1+1)\right|$$

$$D = \left|-\frac{1}{2}\right| + \left|\frac{1}{4} + 2\right| = \boxed{\frac{11}{4} \text{ m}}$$

(b) i. $a(t) = v'(t) = \boxed{\sec^2 t - \frac{t}{2} \text{ miles / hr}^2}$

ii. $a(2\pi) = \sec^2(2\pi) - \frac{2\pi}{2} = \boxed{1 - \pi \text{ miles / hr}^2}$

3. (23 pts) Parts (a) and (b) are unrelated.

(a) Find the equations of the tangent and normal lines to the curve $y = x^{3/2} - x^{1/2}$ at $x = 4$.

(b) Find all values of x on the interval $[0, \pi]$ for which the curve $y = \sin^2 x - \sin x$ has a horizontal tangent line.

Solution:

(a) $y'(x) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = x^{-1/2}\left(\frac{3}{2}x - \frac{1}{2}\right) = \frac{x^{-1/2}}{2}(3x-1) = \frac{3x-1}{2\sqrt{x}}$

$y'(4) = \frac{11}{4}$

$y(4) = 4^{3/2} - 4^{1/2} = 8 - 2 = 6$

Tangent line: $\boxed{y - 6 = \frac{11}{4}(x-4)}$

Normal line: $\boxed{y - 6 = -\frac{4}{11}(x-4)}$

(b) $y'(x) = 2\sin x\cos x - \cos x = \cos x(2\sin x - 1) = 0$

$\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$

$2\sin x - 1 = 0 \quad \Rightarrow \quad \sin x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

Therefore, $\boxed{x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}$

4. (22 pts) Parts (a) and (b) are unrelated.

   (a) Determine $f'(x)$ for the function $f(x) = \dfrac{1}{x+1}$ by using the **definition of derivative**.

   You must obtain f' by evaluating the appropriate **limit** to earn credit.

   (b) Find the values of b and c for which the following function $g(x)$ is differentiable at $x = 2$.

$$g(x) = \begin{cases} \dfrac{3}{8}x^3 & , \quad x < 2 \\ -x^2 + bx + c & , \quad x \geq 2 \end{cases}$$

   You do **not** have to explicitly state the one-sided limits that are being evaluated.

Solution:

(a)

$$f'(x) = \lim_{h \to 0} \frac{\dfrac{1}{(x+h)+1} - \dfrac{1}{x+1}}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{x+h+1} - \frac{1}{x+1}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}\right] = \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{(x+h+1)(x+1)}\right]$$

$$= \lim_{h \to 0} \frac{-1}{(x+h+1)(x+1)} = \boxed{-\frac{1}{(x+1)^2}}$$

(b)

$$g'(x) = \begin{cases} \dfrac{9}{8}x^2 & , \quad x < 2 \\ -2x + b & , \quad x > 2 \end{cases}$$

In order for g to be differentiable at $x = 2$, we must have $\lim_{x \to 2^-} g'(x) = \lim_{x \to 2^+} g'(x)$, which leads to the following:

$$\frac{9}{8}(2^2) = (-2)(2) + b$$

$$\frac{9}{2} = -4 + b$$

$$b = \boxed{\frac{17}{2}}$$

In order for g to be differentiable at $x = 2$, g must also be continuous at $x = 2$.

$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = g(2)$ leads to the following:

$$\frac{3}{8}\left(2^3\right) = -(2^2) + \frac{17}{2}(2) + c$$

$$3 = -4 + 17 + c$$

$$c = \boxed{-10}$$