1. (20 pts) Parts (a) and (b) are not related.

   (a) For $f(x) = \dfrac{1}{x-1}$ and $g(x) = \sqrt{2-x}$, identify the composite function $(f \circ g)(x)$ and its domain. Express the domain in interval form.

   Solution:

   $$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{2-x}\,\right) = \boxed{\dfrac{1}{\sqrt{2-x}-1}}$$

   The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

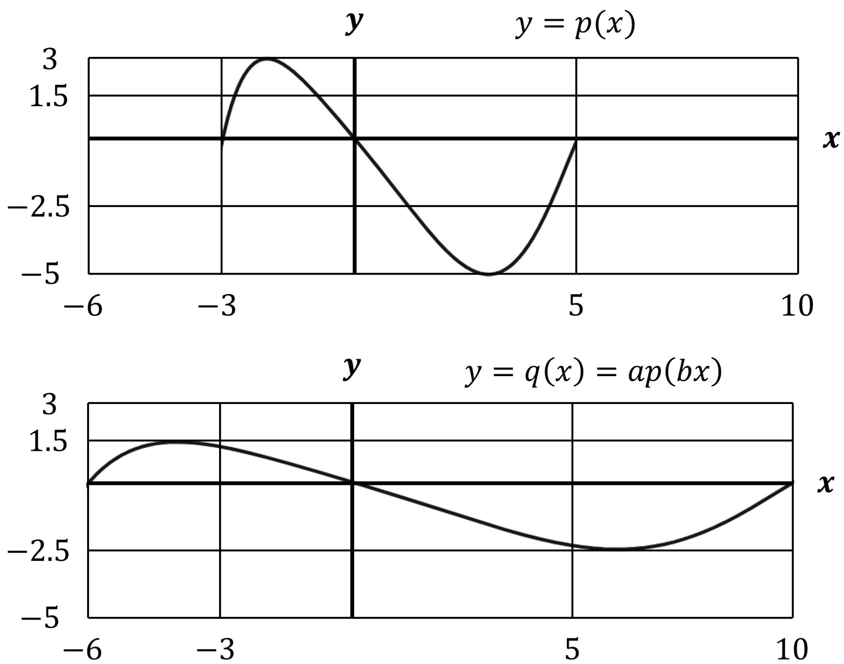
   Domain of g: $\quad 2 - x \geq 0 \quad \Rightarrow \quad x \leq 2$

   Domain of $f(g(x))$:

   $$\begin{aligned} \sqrt{2-x} - 1 &\neq 0 \\ \sqrt{2-x} &\neq 1 \\ 2 - x &\neq 1 \\ x &\neq 1 \end{aligned}$$

   Therefore, the domain of $f \circ g$ is the set of all x such that $x \leq 2$ and $x \neq 1$, which can be written in interval form as $\boxed{(-\infty, 1) \cup (1, 2]}$

(b) The graphs below depict the functions $y = p(x)$ and $y = q(x)$, where q is a transformation of p of the form $q(x) = a\,p(bx)$. Find the values of a and b.



Solution:

The vertical difference between the maximum and minimum values of the curve for $p(x)$ is $3 - (-5) = 8$, while the vertical difference between the maximum and minimum values of the curve for $q(x)$ is $1.5 - (-2.5) = 4$. Therefore, the curve for $q(x)$ has been constructed by applying a vertical contraction of a factor of 2 to the curve for $p(x)$. This implies that $\boxed{a = 1/2}$

The horizontal difference between the endpoints of the curve for $p(x)$ is $5 - (-3) = 8$, while the horizontal difference between the endpoints of the curve for $q(x)$ is $10 - (-6) = 16$. Therefore, the curve for $q(x)$ has been constructed by applying a horizontal expansion of a factor of 2 to the curve for $p(x)$. This implies that $\boxed{b = 1/2}$

Note that $q(x) = 0.5\,p(0.5x)$.

2. (30 pts) Evaluate the following limits. Support your answers by stating theorems, definitions, or other key properties that are used.

   (a) $\lim_{x \to 0} \dfrac{\sin(5x)}{x^2 + 2x}$

Solution: Key property: $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin(5x)}{x^2 + 2x} &= \lim_{x \to 0} \frac{\sin(5x)}{x(x+2)} \\
&= \lim_{x \to 0} \left[\frac{\sin(5x)}{x}\right]\left[\frac{1}{x+2}\right] \\
&= \lim_{x \to 0} \left[\frac{5\sin(5x)}{5x}\right]\left[\frac{1}{x+2}\right] \\
&= \lim_{x \to 0} \left[\frac{\sin(5x)}{5x}\right]\left[\frac{5}{x+2}\right] \\
&= \left[\lim_{x \to 0} \frac{\sin(5x)}{5x}\right]\left[\lim_{x \to 0} \frac{5}{x+2}\right] \\
&= \left[\lim_{x \to 0} \frac{\sin(5x)}{5x}\right]\left[\frac{5}{0+2}\right]
\end{aligned}$$

Let $\theta = 5x$. It follows that $\theta \to 0$ as $x \to 0$.

$$\lim_{x \to 0} \frac{\sin(5x)}{x^2 + 2x} = \frac{5}{2} \lim_{x \to 0} \frac{\sin(5x)}{5x} = \frac{5}{2} \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \left(\frac{5}{2}\right)(1) = \boxed{\frac{5}{2}}$$

(b) $\lim_{x\to 2} \frac{\sqrt{x+1}-\sqrt{3}}{x^2+x-6}$

Solution:

Begin by multiplying the numerator and the denominator by the conjugate of the original numerator expression.

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x+1}-\sqrt{3}}{x^2+x-6} &= \lim_{x\to 2} \frac{\sqrt{x+1}-\sqrt{3}}{x^2+x-6} \cdot \frac{\sqrt{x+1}+\sqrt{3}}{\sqrt{x+1}+\sqrt{3}} \\
&= \lim_{x\to 2} \frac{(\sqrt{x+1})^2-(\sqrt{3})^2}{(x-2)(x+3)(\sqrt{x+1}+\sqrt{3})} \\
&= \lim_{x\to 2} \frac{(x+1)-3}{(x-2)(x+3)(\sqrt{x+1}+\sqrt{3})} \\
&= \lim_{x\to 2} \frac{(x-2)}{(x-2)(x+3)(\sqrt{x+1}+\sqrt{3})} \\
&= \lim_{x\to 2} \frac{1}{(x+3)(\sqrt{x+1}+\sqrt{3})} \\
&= \frac{1}{(2+3)(\sqrt{2+1}+\sqrt{3})} = \frac{1}{(5)(2\sqrt{3})} = \boxed{\frac{1}{10\sqrt{3}}}
\end{aligned}$$

(c) $\lim_{x\to 0} x^4 \cos\left(\frac{1}{2x}\right)$

Solution:

$$-1 \leq \cos\left(\frac{1}{2x}\right) \leq 1$$

$$-x^4 \leq x^4 \cos\left(\frac{1}{2x}\right) \leq x^4$$ (Since x^4 is nonnegative for all x, the inequalities do not reverse direction)

$$\lim_{x\to 0}(-x^4) = \lim_{x\to 0} x^4 = 0$$

Therefore, the Squeeze Theorem indicates that $\lim_{x\to 0} x^4 \cos\left(\frac{1}{2x}\right) = \boxed{0}$

3. (30 pts) Consider the rational function $r(x) = -\dfrac{3x^2+21x+30}{x^2+2x-15}$.

(a) Identify all values of x at which $r(x)$ is discontinuous. At each such x value, explain why the function is discontinuous there.

Solution:

$$r(x) = -\frac{3x^2+21x+30}{x^2+2x-15} = -\frac{3(x^2+7x+10)}{x^2+2x-15} = -\frac{3(x+2)(x+5)}{(x-3)(x+5)}$$

Since $r(x)$ is a rational function, it is continuous at all x in its domain.

Therefore, $r(x)$ is discontinuous only at $\boxed{x = -5, 3}$

(b) Identify the type of discontinuity associated with each x value identified in part (a). Support those classifications by evaluating the appropriate limits.

Solution:

$$r(x) = -\frac{3(x+2)(x+5)}{(x-3)(x+5)} = -\frac{3(x+2)}{x-3}, \quad x \neq -5, 3$$

$$\lim_{x\to -5} r(x) = \lim_{x\to -5}\left[-\frac{3(x+2)}{x-3}\right] = -\frac{3(-5+2)}{-5-3} = -\frac{(3)(-3)}{-8} = -\frac{9}{8}$$

Since the two-sided limit is finite, there is a $\boxed{\text{removable discontinuity at } x = -5}$

$$\lim_{x\to 3^-} r(x) = \lim_{x\to 3^-}\left[-\frac{3(x+2)}{x-3}\right] \to -\frac{15}{0^-} = \infty$$

$$\lim_{x\to 3^+} r(x) = \lim_{x\to 3^+}\left[-\frac{3(x+2)}{x-3}\right] \to -\frac{15}{0^+} = -\infty$$

Since at least one of the two preceding one-sided limits is infinite, there is an $\boxed{\text{infinite discontinuity at } x = 3}$

(c) Find the equation of each vertical asymptote of $y = r(x)$, if any exist. Support your answer in terms of your work in part (b).

Solution:

The finite value of $\lim_{x \to -5} r(x) = -\frac{9}{8}$ determined in part (b) indicates that there is no vertical asymptote at $x = -5$.

The infinite limits $\lim_{x \to 3^-} r(x) = \infty$ and $\lim_{x \to 3^+} r(x) = -\infty$ were determined in part (b). Either one of those limits being infinite is sufficient to conclude that the line $\boxed{x = 3}$ is a vertical asymptote of the curve $y = r(x)$.

(d) Find the equation of each horizontal asymptote of $y = r(x)$, if any exist. Support your answer by evaluating the appropriate limits. *(Reminder: You may not use L'Hôpital's Rule or dominance of powers arguments to evaluate limits on this exam.)*

Solution:

$$\lim_{x \to \pm\infty} r(x) = \lim_{x \to \pm\infty} \left[-\frac{3x^2 + 21x + 30}{x^2 + 2x - 15} \right] = \lim_{x \to \pm\infty} \left[-\frac{3x^2 + 21x + 30}{x^2 + 2x - 15} \right] \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x \to \pm\infty} \left[-\frac{3 + 21/x + 30/x^2}{1 + 2/x - 15/x^2} \right] = -\frac{3 + 0 + 0}{1 + 0 - 0} = -3$$

Therefore, the equation of the only horizontal asymptote is $\boxed{y = -3}$

4. (20 pts) Parts (a) and (b) are not related.

(a) For what value of a is the following function $u(x)$ continuous at $x = 4$? Support your answer using the definition of continuity, which includes evaluating the appropriate limits.

$$u(x) = \begin{cases} \dfrac{x-4}{x^2-16} & , \quad x < 4 \\[2ex] \dfrac{1}{a-x} & , \quad x \geq 4 \end{cases}$$

Solution:

By the definition of continuity, $u(x)$ is continuous at $x = 4$ if $\lim_{x \to 4^-} u(x) = \lim_{x \to 4^+} u(x) = u(4)$.

$$\lim_{x \to 4^-} u(x) = \lim_{x \to 4^-} \frac{x-4}{x^2-16} = \lim_{x \to 4^-} \frac{x-4}{(x-4)(x+4)} = \lim_{x \to 4^-} \frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$$

$$\lim_{x \to 4^+} u(x) = \lim_{x \to 4^+} \frac{1}{a-x} = \frac{1}{a-4}$$

$$u(4) = \frac{1}{a-4}$$

Therefore, $u(x)$ is continuous at $x = 4$ if $\dfrac{1}{8} = \dfrac{1}{a-4}$, which occurs when $\boxed{a = 12}$

(b) Use the Intermediate Value Theorem to establish that the equation $v(x) = x - 2\cos x = 0$ has at least one solution on the interval $(0, \pi/3)$. Verify that all conditions for applying the IVT to this particular problem are satisfied prior to using it.

Solution:

$v(x)$ is continuous for all real numbers x, so $v(x)$ is continuous on the interval $[0, \pi/3]$.

$v(0) = 0 - 2\cos(0) = -2$

$v(\pi/3) = \pi/3 - 2\cos(\pi/3) = \pi/3 - (2)(1/2) = \pi/3 - 1$

So, $v(0) < 0 < v(\pi/3)$. Note that $\pi/3 > 1$ because $\pi > 3$.

Therefore, the Intermediate Value Theorem can be applied to establish that there exists a number c in $(0, \pi/3)$ such that $f(c) = 0$. Such a number c is a solution to the original equation.