APPM 1340
Exam 1
Fall 2023

Name		
Instructor	Richard McNamara	**Section** 150

This exam is worth 100 points and has **4 problems**.

Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to **make a note** indicating the page number where the work is continued or it will **not** be graded.

Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.

Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

There is a **FORMULA SHEET** on the **LAST PAGE** of this exam

End-of-Exam Checklist

1. If you finish the exam before 7:45 PM:
   - Go to the designated area to scan and upload your exam to Gradescope.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.
2. If you finish the exam after 7:45 PM:
   - Please wait in your seat until 8:00 PM.
   - When instructed to do so, scan and upload your exam to Gradescope at your seat.
   - Verify that your exam has been correctly uploaded and all problems have been labeled.
   - Leave the physical copy of the exam with your proctors.

1. (29 pts) Parts (a) - (d) are not related.

   (a) Factor the expression $x^3 + 3x^2 - 4x - 12$.

   Solution:

   Method 1:

$$x^3 + 3x^2 - 4x - 12 = (x^3 + 3x^2) - (4x + 12) = x^2(x+3) - 4(x+3)$$

$$= (x^2 - 4)(x+3) = \boxed{(x-2)(x+2)(x+3)}$$

   Method 2:

$$x^3 + 3x^2 - 4x - 12 = (x^3 - 4x) + (3x^2 - 12) = x(x^2 - 4) + 3(x^2 - 4)$$

$$= (x+3)(x^2 - 4) = \boxed{(x+3)(x-2)(x+2)}$$

   (b) Consider the function $f(x) = (3 - \sqrt{x+1})(3 + \sqrt{x+1})$

      i. What is the domain of $f(x)$? Express your answer using interval notation.

      Solution:

      In order to avoid taking the square root of a negative number, we must have $x + 1 \geq 0$.

      Subtracting 1 from each side leads to $x \geq -1$, which in interval form is $\boxed{[-1, \infty)}$

      ii. Expand $f(x)$ to express it as a polynomial.

      Solution: Using the FOIL method produces a difference of squares.

      $(3 - \sqrt{x+1})(3 + \sqrt{x+1}) = 3^2 - (\sqrt{x+1})^2 = 9 - (x+1) = \boxed{8 - x}$

(c) Solve the following equation for y in terms of x: $\dfrac{5+y}{y} = 3x$

Solution:

$$5 + y = 3xy$$

$$5 = 3xy - y$$

$$(3x - 1)y = 5$$

Therefore, $\boxed{y = \dfrac{5}{3x-1}}$

(d) Find all solutions, if any, of the equation $3x^{1/2} = 2x^{3/2}$

Solution:

$$3x^{1/2} = 2x^{3/2}$$

$$3x^{1/2} - 2x^{3/2} = 0$$

$$x^{1/2}(3 - 2x) = 0$$

The Zero Factor Theorem indicates that $x^{1/2} = 0$ or $3 - 2x = 0$.

$x^{1/2} = 0 \quad \Rightarrow \quad x = 0$

$3 - 2x = 0 \quad \Rightarrow \quad 2x = 3 \quad \Rightarrow \quad x = 3/2$

Therefore, the solution is $\boxed{x = 0, \dfrac{3}{2}}$

2. (28 pts) For the following, let point A be $(-5, 1)$, let point B be $(3, 7)$, let segment AB be the line segment connecting points A and B, and let point M be the midpoint of segment AB.

(a) Find the (x, y) coordinates of point M.

Solution:

The general expression for the midpoint of the line segment connecting the points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Substituting the coordinate values of points A and B leads to the following midpoint of segment AB:

$$\left(\frac{-5 + 3}{2}, \frac{1 + 7}{2}\right) = \boxed{(-1, 4)}$$

(b) Find an equation of the line that is perpendicular to segment AB and passes through point M.

Solution:

The first step is to determine the slope of segment AB. We'll let m_1 denote that slope.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 1}{3 - (-5)} = \frac{6}{8} = \frac{3}{4}$$

The slopes of two perpendicular lines are negative reciprocals of each other. Since the slope of segment AB is $3/4$, the slope of a line that is perpendicular to the segment AB is $-4/3$.

The point-slope form of the equation of the line passing through a point (x_0, y_0) is $y - y_0 = m(x - x_0)$.

Since point M lies on the line that is perpendicular to segment AB at point M, we'll use $(x_0, y_0) = (-1, 4)$, which are the coordinates of point M derived in part (a).

Therefore, an equation for the line that is perpendicular to segment AB and passes through point M is

$y - 4 = -\frac{4}{3}(x - (-1))$, which can be written as $\boxed{y - 4 = -\frac{4}{3}(x + 1)}$

(c) Find the length of segment AB.

Solution:

The length of segment AB is the distance between points A and B, so we'll apply the distance formula.

$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3 - (-5))^2 + (7 - 1)^2} = \sqrt{8^2 + 6^2}$

$D = \sqrt{100} = \boxed{10}$

(d) Find an equation of the circle that is centered at point M and passes through points A and B.

Solution:

The equation of a circle centered at a point (h, k) with a radius of r is $(x - h)^2 + (y - k)^2 = r^2$.

In this case, the circle is centered at point M so that $h = -1$ and $k = 4$, according to the coordinates of point M that were determined in part (a).

Since segment AB represents the diameter of the circle, the circle's radius is half the length of segment AB. The length of segment AB was determined to be 10 in part (c), so that $r = 10/2 = 5$.

Substituting the values of h, k, and r into the general equation for a circle yields $(x - (-1))^2 + (y - 4)^2 = 5^2$.

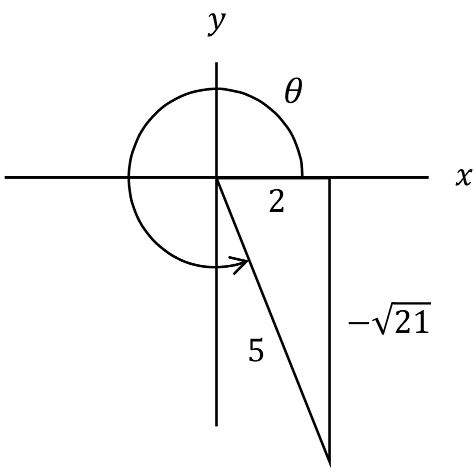
Therefore, the equation of the circle can be written as $\boxed{(x + 1)^2 + (y - 4)^2 = 25}$

3. (18 pts) The following problems are not related.

(a) If $\sec\theta = 5/2$ and θ is on the interval $(3\pi/2, 2\pi)$, find the value of $\sin\theta$.

Solution:

Since θ is on the interval $(3\pi/2, 2\pi)$, there is a triangle in quadrant IV that is associated with angle θ, as shown in the following figure.



Since $\sec\theta = 5/2$, the ratio of the length of the hypotenuse of the triangle to the *adjacent* leg of the triangle is $5/2$ (where *adjacent* refers to the leg that is adjacent to the reference angle). Therefore the hypotenuse and the adjacent leg have been labled with values of 5 and 2, respectively, in the preceding figure.

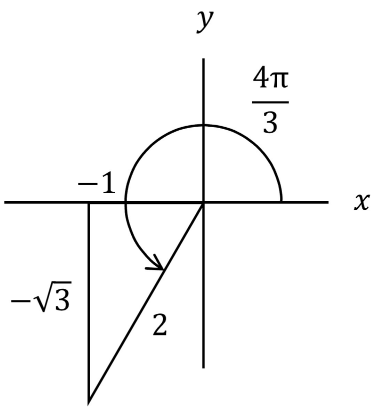
The length of the *opposite* leg in the preceding figure was determined from the Pythagorean Theorem to be $\sqrt{5^2 - 2^2} = \sqrt{21}$. Since the y coordinate associated with the triangle is negative, the opposite leg has been labeled as $-\sqrt{21}$ in the figure.

It follows from the figure that $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{-\sqrt{21}}{5} = \boxed{-\dfrac{\sqrt{21}}{5}}$

(b) Evaluate $\tan\left(\frac{4\pi}{3}\right)$

Solution:

Since $\pi < \frac{4\pi}{3} < \frac{3\pi}{2}$, the angle $\frac{4\pi}{3}$ lies in Quadrant III, as drawn in the following figure.



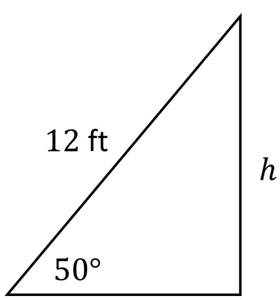
The reference angle is $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$, which is a $60°$ angle in the special $30° - 60° - 90°$ right triangle. The dimensions of such a triangle are proportional to 1, $\sqrt{3}$, and 2, which leads to the set of dimensions displayed in the figure, with the negative signs accounting for the fact that the triangle is in Quadrant III.

It follows from the figure that $\tan\left(\frac{4\pi}{3}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{-\sqrt{3}}{-1} = \boxed{\sqrt{3}}$

(c) One end of a 12-foot rope is anchored to the ground and the other end is tied to the top of a vertical pole. If the rope has been pulled tight and it makes a $50°$ angle with respect to the ground, what is the height of the pole? Your answer should include a trigonometric function term and the correct unit of measurement.

Solution:

The situation is depicted in the following figure:



According to the figure, $\sin 50° = \frac{h}{12}$.

Therefore, $\boxed{h = 12\sin 50° \text{ ft}}$

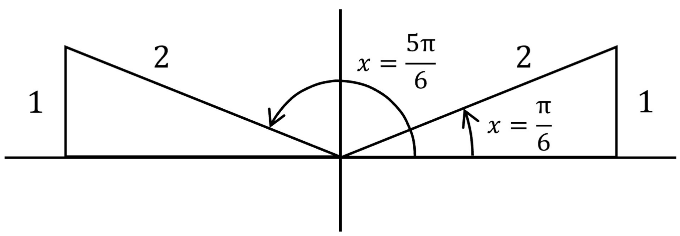
4. (25 pts) The following problems are not related.

(a) Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality $\sin x < 1/2$. Express your answer using interval notation.

Solution:

First, we'll identify the values of x on $[0, 2\pi]$ for which $\sin x = 1/2$. The two triangles depicted in the following figure are associated with

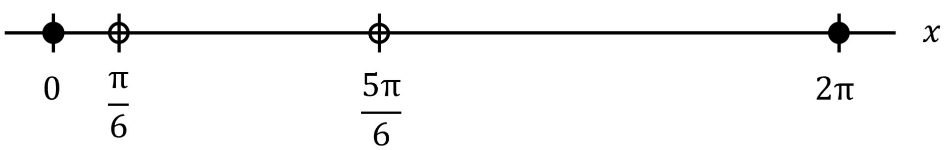
$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}.$$



Both triangles in the preceding figure are associated with an angle whose sine is $1/2$. The leg of length 1 and the hypotenuse of length 2 together imply that both triangles are special $30° - 60° - 90°$ right triangles. In such a triangle, the angle opposite the leg of length 1 is $30°$, which is $\pi/6$ radians. Therefore, the two solutions to the equation $\sin x = 1/2$ are:

$$x = \frac{\pi}{6} \quad \text{and} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

The x values of $\pi/6$ and $5\pi/6$ divide the original interval $[0, 2\pi]$ into three subintervals, as depicted below.



The x values of 0 and 2π are represented by closed circles since those two values are included in the interval provided in the problem statement. The x values of $\pi/6$ and $5\pi/6$ are represented by open circles because values of x for which $\sin x = 1/2$ do not satisfy the requirement that $\sin x < 1/2$.

We can use one value of x from each of the three subintervals to represent those subintervals, as follows:

$x = 0$: $\sin 0 = 0 < 1/2$, which satisfies the specified inequality

$x = \frac{\pi}{2}$: $\sin \frac{\pi}{2} = 1 > 1/2$, which does not satisfy the specified inequality

$x = 2\pi$: $\sin(2\pi) = 0 < 1/2$, which satisfies the specified inequality

Therefore, $\sin x < 1/2$ is satisfied on the interval $\boxed{[0, \pi/6) \cup (5\pi/6, 2\pi]}$

(b) Find the length of the arc that is subtended by an angle of $36°$ on a circle of radius 20 ft. Include the correct unit of measurement.

Solution:

$L = r\theta$

$\theta = (36°)\left(\dfrac{\pi \text{ radians}}{180°}\right) = \dfrac{\pi}{5}$ radians

Therefore, $L = (20 \text{ ft})\left(\dfrac{\pi}{5}\right) = \boxed{4\pi \text{ ft}}$

(c) Is the function $g(x) = (x-1)^2$ odd, even, or neither? Justify your answer by using the definition of odd and/or even functions.

Solution:

$g(-x) = (-x-1)^2 = [-(x+1)]^2 = (x+1)^2$

Since $(x+1)^2 \neq (x-1)^2$, then $g(-x) \neq g(x)$, which means that $g(x)$ is not an even function.

Since $(x+1)^2 \neq -(x-1)^2$, then $g(-x) \neq -g(x)$, which means that $g(x)$ is not an odd function.

Therefore, $\boxed{g(x) \text{ is neither even nor odd}}$

END OF TEST

Your Initials _________

ADDITIONAL BLANK SPACE

If you write a solution here, please clearly indicate the problem number.

Potentially Useful Formulas

Sector of a circle:

Arc length: $L = \theta r$

Area: $A = \frac{1}{2}\theta r^2$

Pythagorean identities:

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Sums and differences:

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Double-angle formulas:

$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$