1. (28 pts) The position function of a particle is given by $s(t) = -10t^2 + 40t + 50$ on the interval $[0, 5]$, where t is measured in seconds and s is measured in feet.

   (a) i. Find all critical numbers of $s(t)$ on the given interval and the corresponding function values.

   Solution:

   $s'(t) = -20t + 40$

   Critical numbers are values of t such that $s'(t) = 0$ or $s'(t)$ does not exist. There are no critical numbers of the latter type for this function.

   $s'(t) = -20t + 40 = 0 \quad \Rightarrow \quad t = 2$

   $\boxed{t = 2}$ is the only critical number of this function, and it does lie on the given interval.

   $s(2) = (-10)(2^2) + (40)(2) + 50 = -40 + 80 + 50 = \boxed{90}$

   ii. Identify the absolute maximum and minimum values of $s(t)$ on the given interval and the corresponding values of t at which they occur.

   Solution:

   The Closed Interval Method compares the function values at the critical numbers and the boundaries of the interval. From part (i) we have the function value at the only critical number:

   $s(2) = 90$

   The function values at the boundaries are as follows:

   $s(0) = (-10)(0^2) + (40)(0) + 50 = 0 + 0 + 50 = 50$
   $s(5) = (-10)(5^2) + (40)(5) + 50 = -250 + 200 + 50 = 0$

   A comparison of the three preceding function values leads to the following abosolute maximum and minimum values of $s(t)$ on $[0, 5]$:

   Absolute maximum: $\boxed{s(2) = 90}$

   Absolute minimum: $\boxed{s(5) = 0}$

(b) i. Determine the particle's velocity and acceleration at $t = 3$ seconds. Include the correct units of measurement.

Solution:

$v(t) = s'(t) = -20t + 40$ (from part (a)(i))

$v(3) = (-20)(3) + 40 = \boxed{-20 \text{ ft/s}}$

$a(t) = v'(t) = -20$

$a(3) = -20 = \boxed{-20 \text{ ft/s}^2}$

ii. Determine the total distance traveled by the particle between $t = 0$ and $t = 5$ seconds. Include the correct unit of measurement.

Solution:

The particle moves in the positive direction when $v(t) > 0$ and it moves in the negative direction when $v(t) < 0$. Since $v(t)$ changes sign at $t = 2$, we need to calculate the distances traveled during the time intervals $[0, 2]$ and $[2, 5]$ separately, and add those results together.

Distance traveled for time interval $[0, 2] = |s(2) - s(0)| = |90 - 50| = 40$

Distance traveled for time interval $[2, 5] = |s(5) - s(2)| = |0 - 90| = 90$

Therefore, the total distance traveled between $t = 0$ and $t = 5$ seconds is $40 + 90 = \boxed{130 \text{ ft}}$

2. (24 pts) Parts (a) and (b) are not related.

(a) Find the equations of the tangent and normal lines to the curve $5y + \sin y + 1 = x^2$ at the point $(-1, 0)$.

Solution:

Use implicit differentiation.

$$\frac{d}{dx}\left[5y + \sin y + 1\right] = \frac{d}{dx}\left[x^2\right]$$

$$5 \cdot \frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = 2x$$

$$(5 + \cos y)\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{5 + \cos y}$$

$$\left.\frac{dy}{dx}\right|_{(-1,0)} = \frac{(2)(-1)}{5 + \cos(0)} = -\frac{2}{5+1} = -\frac{1}{3}$$

Tangent line: $\boxed{y = -\frac{1}{3}(x+1)}$

Normal line: $\boxed{y = 3(x+1)}$

(b) Evaluate $\frac{d}{dx}[x \tan^4 x]$.

Solution:

Use product rule and chain rule.

$$\frac{d}{dx}[x \tan^4 x] = (x)[4\tan^3 x \cdot \sec^2 x] + \left(\tan^4 x\right)(1) = \boxed{\tan^3 x(4x\sec^2 x + \tan x)}$$

3. (35 pts) A 10-foot ladder is initally resting against a vertical wall. Suppose the bottom of the ladder slides away from the wall at a constant rate of 2 feet per second. Determine the values of the following quantities when the top of the ladder is 6 feet above the floor. Include the correct units of measurement.

(a) The speed at which the top of the ladder is sliding down the wall

Solution:

Figure 1 depicts the geometry of the problem, with variables assigned to represent various physical attributes.

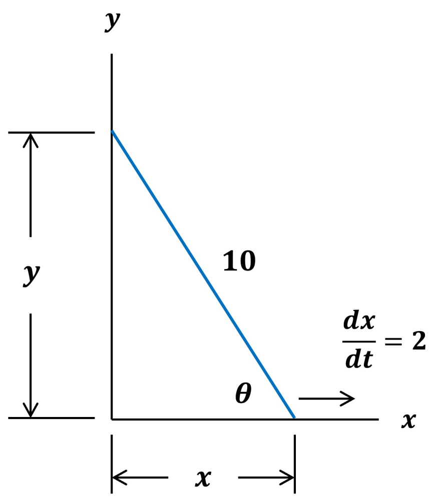

Figure 1

In the context of the assigned variables, dx/dt represents the speed at which the bottom of the ladder is moving across the floor, which is given to be a constant 2 feet per second.

The objective of part (a) involves determining the value of dy/dt when $y = 6$. In order to relate the unknown dy/dt to the known dx/dt, we need a relationship between y and x, which is provided by the Pythagorean Theorem: $x^2 + y^2 = 10^2$.

$$\frac{d}{dt}[x^2 + y^2] = \frac{d}{dt}[100]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

Figure 2 depicts the situation when $y = 6$.

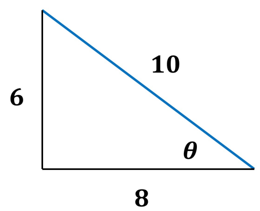

Figure 2

The Pythagorean Theorem has been used to determine that $x = \sqrt{10^2 - 6^2} = 8$.

$$\left.\frac{dy}{dt}\right|_{y=6} = -\left(\frac{8}{6}\right)(2) = -\frac{8}{3}$$

Since dy/dt, which represents the rate of change in y with respect to t, is negative, the speed at which the top of the ladder is sliding *down* the wall is $\boxed{8/3 \text{ ft/s}}$

(b) The rate at which the angle between the ladder and the floor is decreasing

(*Hint:* The distance between the wall and the bottom of the ladder is related to the angle between the ladder and the floor. Start by writing an equation that represents that relationship.)

Solution:

The objective of part (b) involves determining the value of $d\theta/dt$ when $y = 6$. In order to relate the unknown $d\theta/dt$ to the known dx/dt, we need a relationship between θ and x. Figure 1 provides such a relationship:

$$\cos\theta = \frac{x}{10}$$

$$\frac{d}{dt}[\cos\theta] = \frac{d}{dt}\left[\frac{x}{10}\right] \quad \Rightarrow \quad -\sin\theta \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dx}{dt}$$

Figure 2 indicates that $\sin\theta = 6/10$ and we are given that $dx/dt = 2$. It follows that

$$-\frac{6}{10} \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot 2 \quad \Rightarrow \quad \left.\frac{d\theta}{dt}\right|_{y=6} = -\frac{1}{3}$$

Therefore, the rate of decrease of the angle is $\boxed{\frac{1}{3} \text{ rad/s}}$

4. (20 pts) The side of a cube is measured to be 2 cm with a possible error in measurement of up to 0.1 cm.

(a) Identify the function $V(x)$ representing the volume of the cube, where x represents the length of a side.

Solution:

$V(x) = \boxed{x^3}$

(b) Find the linear approximation of $V(x)$ about $x = 2$.

Solution:

$V(x) \approx L(x) = V(2) + V'(2)(x-2)$

$V(2) = 2^3 = 8$

$V'(x) = 3x^2 \quad \Rightarrow \quad V'(2) = (3)(2^2) = 12$

$L(x) = \boxed{8 + 12(x-2)}$

(c) Use differentials to estimate the maximum possible error in computing the volume of the cube. Include the correct units of measurement.

Solution:

$$\frac{dV}{dx} = 3x^2 \quad \Rightarrow \quad dV = 3x^2\,dx$$

$dV\big|_{x=2} = (3)(2^2)(0.1) = \boxed{1.2 \text{ cm}^3}$

5. (15 pts) Verify that the hypotheses of the Mean Value Theorem are satisfied for $g(x) = x + \dfrac{1}{x}$ on the interval $[2, 3]$ and find all numbers c that satisfy the conclusion of the theorem.

Solution:

Polynomials are continuous and differentiable over the set of all real numbers, and rational functions are continuous and differentiable over their entire domains. In particular, $g(x)$ is continuous over all intervals of real numbers that do not include a value of 0. Therefore, both hypotheses of the Mean Value Theorem are satisfied:

$\boxed{g(x) \text{ is continuous on } [2,3]}$ and $\boxed{g(x) \text{ is differentiable on } (2,3)}$

Since the hypotheses of the MVT are satisfied, we know there is at least one real number c on the interval $(2, 3)$ such that

$$g'(c) = \frac{g(3) - g(2)}{3 - 2}$$

$$g(3) = 3 + \frac{1}{3} = \frac{10}{3}$$

$$g(2) = 2 + \frac{1}{2} = \frac{5}{2}$$

$$g'(x) = 1 - \frac{1}{x^2} \quad \Rightarrow \quad g'(c) = 1 - \frac{1}{c^2}$$

Therefore,

$$1 - \frac{1}{c^2} = \frac{10}{3} - \frac{5}{2} = \frac{20 - 15}{6} = \frac{5}{6}$$

$$\frac{1}{6} = \frac{1}{c^2} \quad \Rightarrow \quad c = \pm\sqrt{6}$$

Only the positive root lies on the interval $(2, 3)$. Therefore, the only value of c that satisfies the conclusion of the MVT is $\boxed{c = \sqrt{6}}$

6. (28 pts) Parts (a) and (b) are not related.

(a) Determine $f'(x)$ for the function $f(x) = 1/x$ by using the definition of derivative. (You must obtain f' by evaluating the appropriate limit to earn credit.)

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] = \lim_{h\to 0} \frac{-h}{hx(x+h)}$$

$$f'(x) = \lim_{h\to 0}\left[-\frac{1}{x(x+h)}\right] = \boxed{-\frac{1}{x^2}}$$

(b) Consider the rational function $r(x) = \dfrac{5x^2 - 5x - 10}{3x^2 + 3x - 18}$.

i. Find the (x, y) coordinates of every removable discontinuity of $r(x)$, if any exist. Support your answer by evaluating the appropriate limits.

Solution:

$$r(x) = \frac{5x^2 - 5x - 10}{3x^2 + 3x - 18} = \frac{5(x^2 - x - 2)}{3(x^2 + x - 6)} = \frac{5(x-2)(x+1)}{3(x-2)(x+3)}$$

Based on the preceding factorization, $r(2)$ is undefined due to division by zero, and the numerator is also equal to zero. This indicates the possibility of a removable discontinuity at $x = 2$.

$$\lim_{x\to 2} \frac{5(x-2)(x+1)}{3(x-2)(x+3)} = \lim_{x\to 2} \frac{5(x+1)}{3(x+3)} = \frac{5(2+1)}{3(2+3)} = 1$$

Since the limit is finite, there is a removable discontinuity at $x = 2$, and the limiting value represents the y-coordinate of that removable discontinuity.

Therefore, $r(x)$ has a removable discontinuity at the point $\boxed{(2, 1)}$

ii. Find the equation of every vertical asymptote of $y = r(x)$, if any exist. Support your answer by evaluating the appropriate limits.

Solution:

Based on the factorization in part (i), $r(-3)$ is undefined due to division by zero, and the numerator is nonzero. This indicates the possibility of a vertical asymptote at $x = -3$.

$$\lim_{x \to -3^-} r(x) = \lim_{x \to -3^-} \frac{5(x+1)}{3(x+3)} \to \frac{-10}{(3)(0^-)} = \infty$$

$$\lim_{x \to -3^+} r(x) = \lim_{x \to -3^+} \frac{5(x+1)}{3(x+3)} \to \frac{-10}{(3)(0^+)} = -\infty$$

Since at least one of the two preceding one-sided limits is infinite, the line $\boxed{x = -3}$ is a vertical asymptote of the curve $y = r(x)$.

iii. Find the equation of every horizontal asymptote of $y = r(x)$, if any exist. Support your answer by evaluating the appropriate limits.

Solution:

$$\lim_{x \to \pm\infty} r(x) = \lim_{x \to \pm\infty} \frac{5x^2 - 5x - 10}{3x^2 + 3x - 18} = \lim_{x \to \pm\infty} \frac{5x^2 - 5x - 10}{3x^2 + 3x - 18} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x \to \pm\infty} \frac{5 - 5/x - 10/x^2}{3 + 3/x - 18/x^2} = \frac{5 - 0 - 0}{3 + 0 - 0} = \frac{5}{3}$$

Therefore, the equation of the only horizontal asymptote is $\boxed{y = \frac{5}{3}}$