1. (20 pts) Parts (a) and (b) are not related.

(a) For \( f(x) = \frac{1}{x^2} \) and \( g(x) = \frac{1}{\sqrt{x + 2}} \), identify the composite function \((f \circ g)(x)\) and its domain. Express the domain in interval form.

Solution:

\[
(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{\sqrt{x + 2}}\right) = \left(\frac{1}{\sqrt{x + 2}}\right)^{-2} = (\sqrt{x + 2})^2 = x + 2
\]

The domain of \( f \circ g \) is the set of all \( x \) in the domain of \( g \) such that \( g(x) \) is in the domain of \( f \).

Domain of \( f \): \( x + 2 > 0 \) \( \Rightarrow \) \( x > -2 \)

For each \( x \) in the interval \((-2, \infty)\), \( g(x) \) is in the domain of \( f \) (since \( g(x) \neq 0 \) for all \( x \) values).

Therefore, the domain of \( f \circ g \) is \((-2, \infty)\).
(b) The graph below depicts a function of the form $y = h(x) = a \sin (bx) + c$. Determine the values of $a$, $b$, and $c$. (Hint: Consider the transformations from the graph of $y = \sin x$ to the given graph.)

Solution:

Begin with the graph of the relevant base curve, $y = \sin x$:

The profile of the given curve over the interval $[0, \pi]$ is the same as the profile of the $y = \sin x$ curve over the interval $[0, 3\pi]$. Therefore, the given curve has experienced a horizontal compression of a factor of 3 with respect to the $y = \sin x$ curve, which implies that $b = 3$.

The vertical difference between the given curve’s maximum and minimum values is $1 - (-3) = 4$, while the vertical difference between the $y = \sin x$ curve’s maximum and minimum values is $1 - (-1) = 2$. Therefore, the given curve has experienced a vertical expansion of a factor of 2 with respect to the $y = \sin x$ curve, which implies that $a = 2$.

The vertical center of the given curve is $y = -1$ while the vertical center of the $y = \sin x$ curve is $y = 0$. Therefore, the given curve has experienced a downward vertical shift of 1 unit with respect to the $y = \sin x$ curve, which implies that $c = -1$.

Therefore, the function depicted in the given graph is $y = 2 \sin (3x) - 1$. 
2. (30 pts) Evaluate the following limits. Support your answers by stating theorems, definitions, or other key properties that are used.

(a) \( \lim_{x \to 0} \frac{\tan x \sin (2x)}{x^2} \)

**Solution:** Key property: \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \)

\[
\lim_{x \to 0} \frac{\tan x \sin (2x)}{x^2} = \lim_{x \to 0} \left[ \frac{\tan x}{x} \right] \left[ \frac{\sin (2x)}{x} \right] \\
= \lim_{x \to 0} \left[ \frac{\sin x}{x \cos x} \right] \left[ \frac{2 \sin (2x)}{2x} \right] \\
= \lim_{x \to 0} \left[ \frac{\sin x}{x} \right] \lim_{x \to 0} \left[ \frac{1}{\cos x} \right] \lim_{x \to 0} \left[ \frac{2 \sin (2x)}{2x} \right] \\
= \left[ \lim_{x \to 0} \left[ \frac{\sin x}{x} \right] \right] \left[ \lim_{x \to 0} \left[ \frac{1}{\cos x} \right] \right] \left[ \lim_{x \to 0} \left[ \frac{2 \sin (2x)}{2x} \right] \right] \\
= [1] \left[ \lim_{x \to 0} \frac{\sin (2x)}{2x} \right] = 2 \lim_{x \to 0} \frac{\sin (2x)}{2x}
\]

Let \( \theta = 2x \). It follows that \( \theta \to 0 \) as \( x \to 0 \).

\[
\lim_{x \to 0} \frac{\tan x \sin (2x)}{x^2} = 2 \lim_{x \to 0} \frac{\sin (2x)}{2x} = 2 \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = (2)(1) = 2
\]
(b) \( \lim_{x \to 9} \frac{\sqrt{x - 5} - 2}{x - 9} \)

**Solution:**

Begin by multiplying the numerator and the denominator by the conjugate of the original numerator expression.

\[
\lim_{x \to 9} \frac{\sqrt{x - 5} - 2}{x - 9} = \lim_{x \to 9} \frac{\sqrt{x - 5} - 2}{x - 9} \cdot \frac{\sqrt{x - 5} + 2}{\sqrt{x - 5} + 2}
\]
\[
= \lim_{x \to 9} \frac{(\sqrt{x - 5})^2 - 2^2}{(x - 9)(\sqrt{x - 5} + 2)}
\]
\[
= \lim_{x \to 9} \frac{(x - 5) - 4}{(x - 9)(\sqrt{x - 5} + 2)}
\]
\[
= \lim_{x \to 9} \frac{(x - 9)}{(x - 9)(\sqrt{x - 5} + 2)}
\]
\[
= \lim_{x \to 9} \frac{1}{\sqrt{x - 5} + 2} = \frac{1}{(\sqrt{9 - 5} + 2)} = \frac{1}{4}
\]

(c) \( \lim_{x \to 0} x^8 \sin \left( \frac{1}{x^3} \right) \)

**Solution:**

\[-1 \leq \sin \left( \frac{1}{x^3} \right) \leq 1\]
\[-x^8 \leq x^8 \sin \left( \frac{1}{x^3} \right) \leq x^8\] (Since \( x^8 \) is nonnegative for all \( x \), the inequalities do not reverse direction)

\[
\lim_{x \to 0} (-x^8) = \lim_{x \to 0} x^8 = 0
\]

Therefore, the Squeeze Theorem indicates that \( \lim_{x \to 0} x^8 \sin \left( \frac{1}{x^3} \right) = 0 \)
3. (30 pts) Consider the rational function
\[ r(x) = \frac{x^2 - 5x + 4}{2x^2 - 8x + 6}. \]

(a) Identify all values of \( x \) at which \( r(x) \) is discontinuous. At each such \( x \) value, explain why the function is discontinuous there.

**Solution:**

\[ r(x) = \frac{x^2 - 5x + 4}{2x^2 - 8x + 6} = \frac{(x - 1)(x - 4)}{2(x - 1)(x - 3)} \]

Since \( r(x) \) is a rational function, it is continuous at all \( x \) in its domain.

Therefore, \( r(x) \) is discontinuous only at \( x = 1, 3 \)

(b) Identify the type of discontinuity associated with each \( x \) value identified in part (a). Support those classifications by evaluating the appropriate limits.

**Solution:**

\[ r(x) = \frac{(x - 1)(x - 4)}{2(x - 1)(x - 3)} = \frac{x - 4}{2(x - 3)}, \quad x \neq 1, 3 \]

\[ \lim_{x \to 1} r(x) = \lim_{x \to 1} \frac{x - 4}{2(x - 3)} = \frac{1 - 4}{2(1 - 3)} = \frac{-3}{-4} = \frac{3}{4} \]

Since the two-sided limit is finite, there is a removable discontinuity at \( x = 1 \)

\[ \lim_{x \to 3^-} r(x) = \lim_{x \to 3^-} \frac{x - 4}{2(x - 3)} \to \frac{-1}{(2)(0^-)} = \infty \]

\[ \lim_{x \to 3^+} r(x) = \lim_{x \to 3^+} \frac{x - 4}{2(x - 3)} \to \frac{-1}{(2)(0^+)} = -\infty \]

Since at least one of the two preceding one-sided limits is infinite, there is an infinite discontinuity at \( x = 3 \)
(c) Find the equation of each vertical asymptote of \( y = r(x) \), if any exist. Support your answer in terms of the limits you evaluated in part (b).

**Solution:**

The finite value of \( \lim_{x \to 1} r(x) = \frac{3}{4} \) determined in part (b) indicates that there is no vertical asymptote at \( x = 1 \).

The infinite limits \( \lim_{x \to 3^-} r(x) = \infty \) and \( \lim_{x \to 3^+} r(x) = -\infty \) were determined in part (b). Either one of those limits being infinite is sufficient to conclude that the line \( x = 3 \) is a vertical asymptote of the curve \( y = r(x) \).

(d) Find the equation of each horizontal asymptote of \( y = r(x) \), if any exist. Support your answer by evaluating the appropriate limits.

**Solution:**

\[
\lim_{x \to \pm \infty} r(x) = \lim_{x \to \pm \infty} \frac{x^2 - 5x + 4}{2x^2 - 8x + 6} = \lim_{x \to \pm \infty} \frac{x^2 - 5x + 4}{2x^2 - 8x + 6} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to \pm \infty} \frac{1 - 5/x + 4/x^2}{2 - 8/x + 6/x^2} = \frac{1 - 0 + 0}{2 - 0 + 0} = \frac{1}{2}
\]

Therefore, the equation of the only horizontal asymptote is \( y = \frac{1}{2} \).
4. (20 pts) Parts (a) and (b) are not related.

(a) For what value of $b$ is the following function $u(x)$ continuous at $x = 3$? Support your answer using the definition of continuity, which includes evaluating the appropriate limits.

$$u(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x < 3 \\ 5x + b, & x \geq 3 \end{cases}$$

Solution:

By the definition of continuity, $u(x)$ is continuous at $x = 3$ if

$$\lim_{x \to 3^-} u(x) = \lim_{x \to 3^+} u(x) = u(3).$$

$$\lim_{x \to 3^-} u(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3^-} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3^-} (x + 3) = 3 + 3 = 6$$

$$\lim_{x \to 3^+} u(x) = \lim_{x \to 3^+} (5x + b) = (5)(3) + b = 15 + b$$

$$u(3) = (5)(3) + b = 15 + b$$

Therefore, $u(x)$ is continuous at $x = 3$ if $6 = 15 + b$, which occurs when $b = -9$.

(b) The Intermediate Value Theorem can NOT be used to guarantee that $v(x) = \frac{2}{x} + \sqrt{x + 2} = 0$ for a value of $x$ on the interval $(-1, 2)$. Explain which condition for applying the theorem is not satisfied in this case.

Solution:

The Intermediate Value Theorem cannot be applied in this case because $v(0)$ is undefined, which means that $v(x)$ is not continuous on the interval $[-1, 2]$.

The continuity of $v(x)$ on $[-1, 2]$ is one of the hypotheses for applying the IVT to the given function on the given interval.

(Note that $v(-1) = -1$ and $v(2) = 3$ together indicate that the other IVT hypothesis does hold)