1. (20 pts) Parts (a) and (b) are not related.

   (a) For $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{\sqrt{x+2}}$, identify the composite function $(f \circ g)(x)$ and its domain. Express the domain in interval form.

   Solution:

   $$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{\sqrt{x+2}}\right) = \left(\frac{1}{\sqrt{x+2}}\right)^{-2} = (\sqrt{x+2})^2 = \boxed{x+2}$$

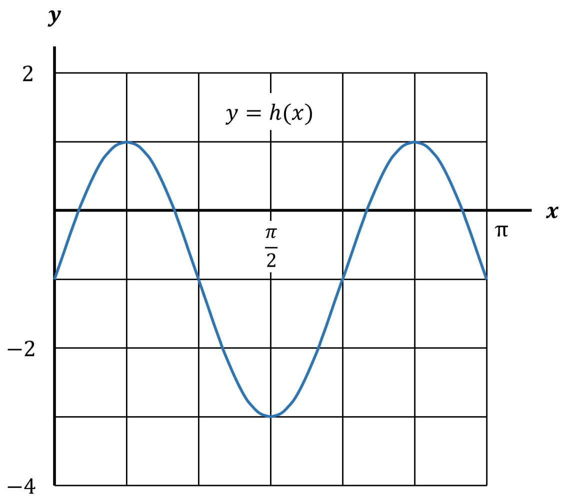
   The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

   Domain of f: $\quad x + 2 > 0 \quad \Rightarrow \quad x > -2$

   For each x in the interval $(-2, \infty)$, $g(x)$ is in the domain of f (since $g(x) \neq 0$ for all x values).

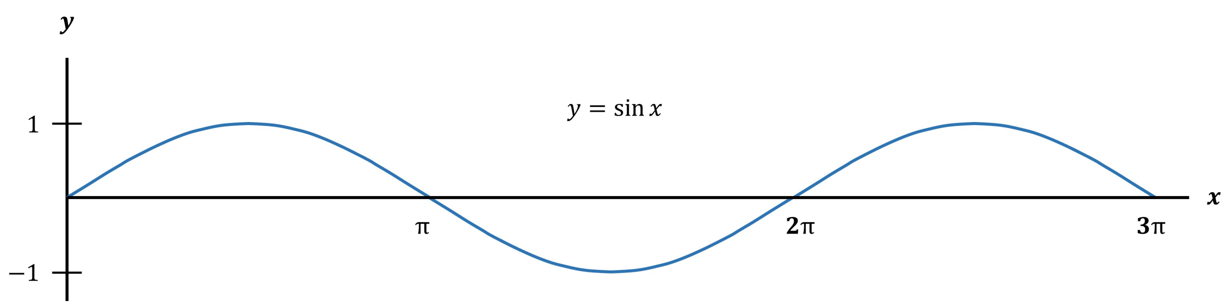
   Therefore, the domain of $f \circ g$ is $\boxed{(-2, \infty)}$

(b) The graph below depicts a function of the form $y = h(x) = a\sin(bx) + c$. Determine the values of a, b, and c. (*Hint:* Consider the transformations from the graph of $y = \sin x$ to the given graph.)



Solution:

Begin with the graph of the relevant base curve, $y = \sin x$:



The profile of the given curve over the interval $[0, \pi]$ is the same as the profile of the $y = \sin x$ curve over the interval $[0, 3\pi]$. Therefore, the given curve has experienced a horizontal compression of a factor of 3 with respect to the $y = \sin x$ curve, which implies that $\boxed{b = 3}$

The vertical difference between the given curve's maximum and minimum values is $1 - (-3) = 4$, while the vertical difference between the $y = \sin x$ curve's maximum and minimum values is $1 - (-1) = 2$. Therefore, the given curve has experienced a vertical expansion of a factor of 2 with respect to the $y = \sin x$ curve, which implies that $\boxed{a = 2}$

The vertical center of the given curve is $y = -1$ while the vertical center of the $y = \sin x$ curve is $y = 0$. Therefore, the given curve has experienced a downward vertical shift of 1 unit with respect to the $y = \sin x$ curve, which implies that $\boxed{c = -1}$

Therefore, the function depicted in the given graph is $\boxed{y = 2\sin(3x) - 1}$

2. (30 pts) Evaluate the following limits. Support your answers by stating theorems, definitions, or other key properties that are used.

   (a) $\displaystyle\lim_{x\to 0} \frac{\tan x \sin(2x)}{x^2}$

Solution: Key property: $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$$
\begin{aligned}
\lim_{x\to 0} \frac{\tan x \sin(2x)}{x^2} &= \lim_{x\to 0} \left[\frac{\tan x}{x}\right]\left[\frac{\sin(2x)}{x}\right] \\
&= \lim_{x\to 0} \left[\frac{\sin x}{x\cos x}\right]\left[\frac{2\sin(2x)}{2x}\right] \\
&= \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right)\right]\left[\frac{2\sin(2x)}{2x}\right] \\
&= \left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\right]\left[\lim_{x\to 0}\left(\frac{2}{\cos x}\right)\right]\left[\lim_{x\to 0}\frac{\sin(2x)}{2x}\right] \\
&= [1]\left[\frac{2}{1}\right]\left[\lim_{x\to 0}\frac{\sin(2x)}{2x}\right] = 2\lim_{x\to 0}\frac{\sin(2x)}{2x}
\end{aligned}
$$

Let $\theta = 2x$. It follows that $\theta \to 0$ as $x \to 0$.

$$\lim_{x\to 0} \frac{\tan x \sin(2x)}{x^2} = 2\lim_{x\to 0}\frac{\sin(2x)}{2x} = 2\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = (2)(1) = \boxed{2}$$

(b) $\lim_{x\to 9} \frac{\sqrt{x-5}-2}{x-9}$

Solution:

Begin by multiplying the numerator and the denominator by the conjugate of the original numerator expression.

$$\begin{aligned}
\lim_{x\to 9} \frac{\sqrt{x-5}-2}{x-9} &= \lim_{x\to 9} \frac{\sqrt{x-5}-2}{x-9} \cdot \frac{\sqrt{x-5}+2}{\sqrt{x-5}+2} \\
&= \lim_{x\to 9} \frac{(\sqrt{x-5})^2 - 2^2}{(x-9)(\sqrt{x-5}+2)} \\
&= \lim_{x\to 9} \frac{(x-5)-4}{(x-9)(\sqrt{x-5}+2)} \\
&= \lim_{x\to 9} \frac{(x-9)}{(x-9)(\sqrt{x-5}+2)} \\
&= \lim_{x\to 9} \frac{1}{(\sqrt{x-5}+2)} = \frac{1}{(\sqrt{9-5}+2)} = \boxed{\frac{1}{4}}
\end{aligned}$$

(c) $\lim_{x\to 0} x^8 \sin\left(\frac{1}{x^3}\right)$

Solution:

$-1 \leq \sin\left(\frac{1}{x^3}\right) \leq 1$

$-x^8 \leq x^8 \sin\left(\frac{1}{x^3}\right) \leq x^8$ (Since x^8 is nonnegative for all x, the inequalities do not reverse direction)

$\lim_{x\to 0}(-x^8) = \lim_{x\to 0} x^8 = 0$

Therefore, the Squeeze Theorem indicates that $\lim_{x\to 0} x^8 \sin\left(\frac{1}{x^3}\right) = \boxed{0}$

3. (30 pts) Consider the rational function $r(x) = \dfrac{x^2 - 5x + 4}{2x^2 - 8x + 6}$.

(a) Identify all values of x at which $r(x)$ is discontinuous. At each such x value, explain why the function is discontinuous there.

Solution:

$$r(x) = \frac{x^2 - 5x + 4}{2x^2 - 8x + 6} = \frac{(x-1)(x-4)}{2(x-1)(x-3)}$$

Since $r(x)$ is a rational function, it is continuous at all x in its domain.

Therefore, $r(x)$ is discontinuous only at $\boxed{x = 1, 3}$

(b) Identify the type of discontinuity associated with each x value identified in part (a). Support those classifications by evaluating the appropriate limits.

Solution:

$$r(x) = \frac{(x-1)(x-4)}{2(x-1)(x-3)} = \frac{(x-4)}{2(x-3)}, \quad x \neq 1, 3$$

$$\lim_{x \to 1} r(x) = \lim_{x \to 1} \frac{x-4}{2(x-3)} = \frac{1-4}{(2)(1-3)} = \frac{-3}{-4} = \frac{3}{4}$$

Since the two-sided limit is finite, there is a $\boxed{\text{removable discontinuity at } x = 1}$

$$\lim_{x \to 3^-} r(x) = \lim_{x \to 3^-} \frac{x-4}{2(x-3)} \to \frac{-1}{(2)(0^-)} = \infty$$

$$\lim_{x \to 3^+} r(x) = \lim_{x \to 3^+} \frac{x-4}{2(x-3)} \to \frac{-1}{(2)(0^+)} = -\infty$$

Since at least one of the two preceding one-sided limits is infinite, there is an $\boxed{\text{infinite discontinuity at } x = 3}$

(c) Find the equation of each vertical asymptote of $y = r(x)$, if any exist. Support your answer in terms of the limits you evaluated in part (b).

Solution:

The finite value of $\lim\limits_{x\to 1} r(x) = \dfrac{3}{4}$ determined in part (b) indicates that there is no vertical asymptote at $x = 1$.

The infinite limits $\lim\limits_{x\to 3^-} r(x) = \infty$ and $\lim\limits_{x\to 3^+} r(x) = -\infty$ were determined in part (b). Either one of those limits being infinite is sufficient to conclude that the line $\boxed{x = 3}$ is a vertical asymptote of the curve $y = r(x)$.

(d) Find the equation of each horizontal asymptote of $y = r(x)$, if any exist. Support your answer by evaluating the appropriate limits.

Solution:

$$\lim_{x\to\pm\infty} r(x) = \lim_{x\to\pm\infty} \frac{x^2 - 5x + 4}{2x^2 - 8x + 6} = \lim_{x\to\pm\infty} \frac{x^2 - 5x + 4}{2x^2 - 8x + 6} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\pm\infty} \frac{1 - 5/x + 4/x^2}{2 - 8/x + 6/x^2} = \frac{1 - 0 + 0}{2 - 0 + 0} = \frac{1}{2}$$

Therefore. the equation of the only horizontal asymptote is $\boxed{y = \dfrac{1}{2}}$

4. (20 pts) Parts (a) and (b) are not related.

(a) For what value of b is the following function $u(x)$ continuous at $x = 3$? Support your answer using the definition of continuity, which includes evaluating the appropriate limits.

$$u(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & , \quad x < 3 \\ 5x + b & , \quad x \geq 3 \end{cases}$$

Solution:

By the definition of continuity, $u(x)$ is continuous at $x = 3$ if $\lim_{x \to 3^-} u(x) = \lim_{x \to 3^+} u(x) = u(3)$.

$$\lim_{x \to 3^-} u(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3^-} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3^-} (x + 3) = 3 + 3 = 6$$

$$\lim_{x \to 3^+} u(x) = \lim_{x \to 3^+} (5x + b) = (5)(3) + b = 15 + b$$

$$u(3) = (5)(3) + b = 15 + b$$

Therefore, $u(x)$ is continuous at $x = 3$ if $6 = 15 + b$, which occurs when $\boxed{b = -9}$

(b) The Intermediate Value Theorem can **NOT** be used to guarantee that $v(x) = \dfrac{2}{x} + \sqrt{x + 2} = 0$ for a value of x on the interval $(-1, 2)$. Explain which condition for applying the theorem is not satisfied in this case.

Solution:

The Intermediate Value Theorem cannot be applied in this case because $v(0)$ is undefined, which means that

$\boxed{v(x) \text{ is not continuous on the interval } [-1, 2]}$

The continuity of $v(x)$ on $[-1, 2]$ is one of the hypotheses for applying the IVT to the given function on the given interval.

(Note that $v(-1) = -1$ and $v(2) = 3$ together indicate that the other IVT hypothesis does hold)