1. (35pts) The following problems are not related.

(a)(16pts) Consider the following problem: *A plane flying horizontally at an altitude of* $1mi$ *and a speed of* $500mi/h$ *passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is* $2mi$ *away from the station.* **Now answer the following questions:** (i)(4pts) Write down the information that is given in this problem (use the notation established in the diagram below). (ii)(4pts) Write down what you are trying to find in this problem. (iii)(8pts) Solve the problem. Simplify your answer.

(b)(16pts) Find the equation of the *tangent line* to $y = \sqrt{x}$ at the point $(1, 1)$. Simplify your answer.

(c)(3pts) Which choice below is the correct *derivative* of $f(x) = \dfrac{x^2 - 2}{2x + 1}$? (**No justification necessary** - *Choose only one answer, copy down the entire answer.*)

(A) $f'(x) = \dfrac{2x}{(2x+1)^2}$ (B) $f'(x) = \dfrac{3x^2 - 3}{(2x+1)^2}$ (C) $f'(x) = \dfrac{2x^2 + 2x + 4}{2x^2 + 4x + 1}$ (D) $f'(x) = \dfrac{2(x^2 + x + 2)}{(2x+1)^2}$ (E) $f'(x) = \dfrac{2x^2 + 2x + 4}{(x+2)^2}$

Solution:

(a)(i)(4pts) We are $\boxed{\text{given } \dfrac{dx}{dt} = 500\,\text{mi/h.}}$

(a)(ii)(4pts) We wish to $\boxed{\text{find } \dfrac{dz}{dt} \text{ when } z = 2\,\text{mi.}}$

(a)(iii)(8pts) Note that

$$z^2 = x^2 + 1^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

and when $z = 2$ we have

$$2^2 = x^2 + 1^2 \Rightarrow x^2 = 2^2 - 1^2 = 4 - 1 = 3 \Rightarrow x = \pm\sqrt{3} \Rightarrow x = \sqrt{3}$$

Thus

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = \frac{\sqrt{3}}{2} \cdot 500 = \boxed{250\sqrt{3}\,\text{mi/h.}}$$

(b)(16pts) Note that $f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2}x^{-1/2} \Rightarrow f'(1) = \dfrac{1}{2}$ thus

$$y = f(1) + f'(1)(x - 1) \;\Rightarrow\; y = 1 + \frac{1}{2}(x - 1) \;\Rightarrow\; \boxed{y = \frac{x}{2} + \frac{1}{2}.}$$

(c)(3pts) $\boxed{\text{Choice D.}}$ By the Quotient Rule, we have

$$\left[\frac{x^2 - 2}{2x + 1}\right]' = \frac{2x \cdot (2x+1) - (x^2 - 2) \cdot 2}{(2x+1)^2} = \frac{4x^2 + 2x - 2x^2 + 4}{(2x+1)^2} = \frac{2x^2 + 2x + 4}{(2x+1)^2} = \boxed{\frac{2(x^2 + x + 2)}{(2x+1)^2}}.$$

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2. (34pts) Start this problem on a **new** page. The following problems are not related.

(a)(17pts) Suppose r represents the radius of a circular disk. (i)(8pts) If $A(r) = \pi r^2$ find dA, the *differential* of A. (ii)(9pts) Suppose the radius of the circle was originally found to be $10\,\text{cm}$ but expands to $10.2\,\text{cm}$, use differentials to estimate the change in the area, ΔA.

(b)(17pts) Find the *absolute minimum* and *absolute maximum values* of $f(x) = (x^2 - 1)^3$ on the interval $[-1, 2]$. Give your answer in the form (x, y). Show all work, justify your answers and clearly label your answers.

Solution:

(a)(i)(8pts) If $A = \pi r^2$ then $\boxed{dA = 2\pi r dr.}$

(a)(ii)(9pts) Note that $\Delta r = 10.2 - 10 = 0.2$ thus

$$\Delta A \approx dA = 2\pi r dr\Big|_{r=10, dr=\Delta r} = 2\pi(10)(0.2) = 4\pi \text{ cm}^2$$

so the $\boxed{\text{area will increase by approximately } 4\pi \text{ cm}^2.}$

(b)(17pts) By the Chain Rule, we have

$$f(x) = (x^2 - 1)^3 \Rightarrow f'(x) = 3(x^2 - 1)^2 \cdot 2x = 6x[(x - 1)(x + 1)]^2 \Rightarrow f'(x) = 0 \quad \Leftrightarrow \quad x = -1, 0, 1$$

Now plugging in the endpoints and critical points into $f(x)$ yields

$$f(-1) = \left((-1)^2 - 1\right)^3 = 0,\ f(0) = \left((0)^2 - 1\right)^3 = -1,\ f(1) = \left((1)^2 - 1\right)^3 = 0 \text{ and } f(2) = \left((2)^2 - 1\right)^3 = 27$$

thus an $\boxed{\text{absolute maximum occurs at } (2, 27) \text{ and and absolute minimum occurs at } (0, -1).}$

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3. (34pts) Start this problem on a **new** page. The following problems are not related.

(a)(17pts) Use the *Squeeze Theorem* to evaluate the following limit: $\lim_{x \to 0^+} \sqrt{x} \cos^2\left(\frac{1}{x}\right)$. Show all work, explain.

(b)(17pts) Suppose $y = f(x)$, use *implicit differentiation* to find y' if $y \cos(x) = x^2 + y^2$.

Solution:

(a)(17pts) Note that for all $x > 0$ we have

$$-1 \le \cos(1/x) \le 1 \ \Rightarrow\ 0 \le \cos^2(1/x) \le 1 \ \Rightarrow\ 0 \le \sqrt{x} \cos^2(1/x) \le \sqrt{x}$$

and, finally, observe that $\lim_{x \to 0^+} 0 = \lim_{x \to 0^+} \sqrt{x} = 0$, thus, by Squeeze Theorem, we have $\lim_{x \to 0^+} \sqrt{x} \cos^2(1/x) = 0$.

(b)(17pts) Differentiating both sides of the equation with respect to x yields

$$y \cos(x) = x^2 + y^2 \overset{d/dx}{\Rightarrow} y' \cos(x) - y \sin(x) = 2x + 2yy' \ \Rightarrow\ y' \cos(x) - 2yy' = 2x + y \sin(x) \ \Rightarrow\ \boxed{y' = \frac{2x + y \sin(x)}{\cos(x) - 2y}.}$$

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4. (35pts) Start this problem on a **new** page. The following problems are not related.

(a)(16pts) Suppose $g(x) = \begin{cases} x^2 + x, & \text{if } x < 0, \\ 1 - \cos(x), & \text{if } x = 0, \\ \sin(x), & \text{if } x > 0. \end{cases}$ (i)(8pts) Find the $\lim_{x \to 0} g(x)$. (ii)(8pts) Show that $g(x)$ is continuous at $x = 0$. Be sure to show that all the conditions of continuity have been satisfied.

(b)(16pts) (i)(8pts) Write down the *piecewise* definition of the function $f(x) = 1 + |x^2 - 4|$.

(ii)(8pts) Find the derivative of $f(x) = 1 + |x^2 - 4|$.

(c)(3pts) The function $h(x) = \dfrac{3x+1}{\sqrt[3]{5+8x^3}}$ has a *horizontal asymptote* at which choice below? (**No justification necessary** - *Choose only <u>one</u> answer, copy down the entire answer.*)

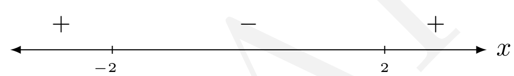
(A) $y=0$ (B) $y=1.5$ (C) $y=0$ and $y=3/2$ (D) $y=-3/2$ and $y=3/2$ (E) None of these

Solution: (a)(i)(8pts) At $x=0$ we have to check the one-side limits, note that

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x^2 + x = 0 \text{ and } \lim_{x\to 0^+} g(x) = \sin(0) = 0 \Rightarrow \lim_{x\to 0} g(x) = 0.$$

(a)(ii)(8pts) $\boxed{\text{Yes, } g(x) \text{ is continuous}}$ since $g(0) = 1-\cos(0) = 1-1 = 0$ so $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = 0 = g(0)$.

(b)(i)(8pts) Note that $x^2-4 = (x-2)(x+2)$ and so we can check the sign of $f(x)$ with a number line



thus we have

$$f(x) = 1 + |x^2-4| = \begin{cases} 1+(x^2-4), & \text{if } x \le -2 \text{ or } x \ge 2, \\ 1-(x^2-4), & \text{if } -2 < x < 2, \end{cases} = \boxed{\begin{cases} x^2-3, & \text{if } x \le -2 \text{ or } x \ge 2, \\ 5-x^2, & \text{if } -2 < x < 2. \end{cases}}$$

(b)(ii)(8pts) Note that

$$f(x) = \begin{cases} x^2-3, & \text{if } x \le -2 \text{ or } x \ge 2, \\ 5-x^2, & \text{if } -2 < x < 2. \end{cases} \Rightarrow f'(x) = \boxed{\begin{cases} 2x, & \text{if } x < -2 \text{ or } x > 2, \\ -2x, & \text{if } -2 < x < 2. \end{cases}}.$$

(c)(3pts) $\boxed{\text{Choice (B).}}$ <u>Discussion:</u> Note that

$$\lim_{x\to\infty} \frac{3x+1}{\sqrt[3]{8x^3+5}} = \lim_{x\to\infty} \frac{x\cdot(3+1/x)}{x\cdot\sqrt[3]{8+5/x^3}} = \frac{3}{\sqrt[3]{8}} = \frac{3}{2} \text{ and } \lim_{x\to-\infty} \frac{3x+1}{\sqrt[3]{8x^3+5}} \overset{DOP}{=} \frac{3}{2} = 1.5$$

thus we see that $y=1.5$ is the only horizontal asymptote which implies choice (B).

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5. (12pts) Answer either **ALWAYS TRUE** or **FALSE**. You do <u>NOT</u> need to justify your answer. (*Don't just write down "A.T." or "F", completely write out the words "ALWAYS TRUE" or "FALSE" depending on your answer.*)

(a)(3pts) The Mean Value Theorem applies to $h(x) = -\dfrac{1}{x}$ on $[-3, -\frac{1}{2}]$ and the value of $c \in (-3, -\frac{1}{2})$ that satisfies the Mean Value Theorem is $c = -\frac{2}{3}$.

(b)(3pts) Suppose $f(x)$ is *continuous* for all x, then $f(x)$ is *differentiable* at $x=a$ if $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists.

(c)(3pts) Suppose the position function of a particle (at time $t \ge 0$ in seconds) is given by $s(t) = t^2 - t$ meters, then the *total distance* traveled during the time period $0 \le t \le 1$ by the particle is 0.5 meters.

(d)(3pts) The function $f(x) = \dfrac{x-2}{x^3-2x^2}$ has a *vertical asymptote* at $x=0$ and a *jump discontinuity* at $x=2$.

Solution: $\boxed{\text{3pts each: (a) FALSE (b) ALWAYS TRUE (c) ALWAYS TRUE (d) FALSE}}$

<u>Discussion:</u>

(a) False. Here we have the interval $(a, b) = (-3, -1/2)$ and so

$$\frac{h(b) - h(a)}{b - a} = \frac{h(-1/2) - h(-3)}{-1/2 - (-3)} = \frac{2 - 1/3}{3 - 1/2} = \frac{5/3}{5/2} = \frac{2}{3} = \frac{1}{c^2} = h'(c)$$

and so $c = \pm\sqrt{3/2} \neq -2/3 \Rightarrow$ F.

(b) Always True. If $f(x)$ is continuous at $x = a$, then, by definition, $f(x)$ is differentiable at $x = a$ if and only if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists $\Rightarrow$ A.T.

(c) Always True. The total distance travelled is 1/2. Note that $v(t) = s'(t) = 2t - 1$ and $v'(t) = 0$ when $t = 1/2$ thus

$$\text{Total Distance} = |s(0) - s(1/2)| + |s(1) - s(1/2)| = |0 - (-1/4)| + |0 - (-1/4)| = 1/2 \Rightarrow \text{ A.T.}$$

(d) False. Factoring the denominator yields $x^3 - 2x^2 = x^2(x - 2)$, and so we see that there are possible vertical asymptotes at $x = 0, 2$. Taking limits as $x \to 0$ gives

$$\lim_{x \to 0} \frac{x - 2}{x^3 - 2x^2} = \lim_{x \to 0} \frac{x - 2}{x^2(x - 2)} = \lim_{x \to 0} \frac{1}{x^2} \overset{N/0}{=} +\infty$$

thus we have a *infinite discontinuity* (and vertical asymptote) at $x = 0$. However, taking the limit as $x \to 2$ yields the indeterminate form "0/0", and factoring yields

$$\lim_{x \to 2} \frac{x - 2}{x^3 - 2x^2} \overset{\text{"0/0"}}{=} \lim_{x \to 2} \frac{x - 2}{x^2(x - 2)} = \lim_{x \to 2} \frac{1}{x^2} = \frac{1}{4}$$

thus we have a *removable discontinuity* at $x = 2 \Rightarrow$ F.
