1. (24pts) The following problems are not related. Show all work.

(a)(12pts) Use the *Quotient Rule* to find $f'(x)$ if $f(x) = \dfrac{x^2-2}{2x+1}$. Simplify your answer.

(b)(12pts) If $y = \sin\big(\cot(x)\big)$ find dy/dx. Simplify your answer.

Solution:

(a)(12pts) By the Quotient Rule, we have

$$\left[\frac{x^2-2}{2x+1}\right]' = \frac{2x\cdot(2x+1)-(x^2-2)\cdot 2}{(2x+1)^2} = \frac{4x^2+2x-2x^2+4}{(2x+1)^2} = \frac{2x^2+2x+4}{(2x+1)^2}.$$

(b)(12pts) By the Chain Rule, we have

$$\frac{d}{dx}\Big[\sin\big(\cot(x)\big)\Big] = \cos\big(\cot(x)\big)\cdot -\csc^2(x) = -\cos\big(\cot(x)\big)\csc^2(x).$$

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2. (28pts) Start this problem on a **new** page. The following problems are not related.

(a)(12pts) Suppose y is a function of x, use *implicit differentiation* to find y' if $y\cos(x) = x^2 + y^2$.

(b)(12pts) Find the equation of the *tangent line* to $y = \sqrt{x}$ at the point $(1,1)$. Simplify your answer.

(c)(4pts) Which of the choices below is equivalent to the limit $\displaystyle\lim_{x\to 1}\frac{x^4+x-2}{x-1}$? **Choose only <u>one</u> answer.** *No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.*

(A) $4x^3-1$ (B) 5 (C) $\dfrac{0}{0}$ (D) 3 (E) None of these

Solution:

(a)(12pts) Differentiating both sides of the equation with respect to x yields

$$y\cos(x) = x^2+y^2 \overset{d/dx}{\Rightarrow} y'\cos(x) - y\sin(x) = 2x + 2yy' \;\Rightarrow\; y'\cos(x) - 2yy' = 2x + y\sin(x) \;\Rightarrow\; y' = \frac{2x+y\sin(x)}{\cos(x)-2y}.$$

(b)(12pts) Note that $f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2}x^{-1/2} \Rightarrow f'(1) = \dfrac{1}{2}$ thus

$$y = f(1) + f'(1)(x-1) \;\Rightarrow\; y = 1 + \frac{1}{2}(x-1) \;\Rightarrow\; y = \frac{x}{2} + \frac{1}{2}.$$

(c)(4pts) Choice B. Note that if we let $f(x) = x^4 + x$ then, using the limit definition of the derivative, we have

$$\lim_{x\to 1}\frac{x^4+x-2}{x-1} = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = f'(1) = \left(4x^3+1\right)\Big|_{x=1} = 5 \Rightarrow \text{Choice (B)}$$

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3. (20pts) Start this problem on a **new** page. The following problems are not related.

(a)(10pts) For what value(s) of $x \in \mathbb{R}$ does the function $f(x) = 2x^3 + 3x^2 - 12x + 1$ have a *horizontal tangent*?

(b)(10pts) The position function of a particle is given by $s(t) = t^3 - 4.5t^2 - 7t$ where $t \geq 0$ is in seconds and distance is in feet. (i)(5pts) Find the velocity of the particle as a function of t. (ii)(5pts) When is the acceleration equal to 0?

Solution:

(a)(10pts) We need to find all x in the domain such that $f'(x) = 0$, note that

$$f'(x) = [2x^3 + 3x^2 - 12x + 1]' = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1)$$

thus $f'(x) = 0 \Rightarrow x = -2, 1$ which is in the domain since $f(x)$ is a polynomial thus $f(x)$ has horizontal tangents at $x = -2, 1$.

(b)(i)(5pts) Here we have the velocity is $v(t) = s'(t) = [t^3 - 4.5t^2 - 7t]' = 3t^2 - 9t - 7$.

(b)(ii)(5pts) The acceleration is $a(t) = v'(t) = [3t^2 - 9t - 7]' = 6t - 9$ thus $a(t) = 0 \Rightarrow 6t - 9 = 0 \Rightarrow t = \frac{9}{6}$ sec.

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4. (28pts) Start this problem on a **new** page. The following problems are not related.

(a)(12pts) If $y = \sec(x)$, find y''. Show all work.

(b)(12pts) For what values of a and b will the function $f(x) = \begin{cases} x^2 - 3x, & \text{if } x < 2 \\ ax^2 + b, & \text{if } x \geq 2 \end{cases}$ be differentiable at $x = 2$? Explain.

(c)(4pts) If $h(x) = \sqrt{4 + 3f(x)}$ where $f(1) = 7$ and $f'(1) = 4$, then $h'(1)$ is equal to which choice below? **Choose only <u>one</u> answer.** *No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.*

(A) 5 (B) $\dfrac{25}{2}$ (C) 10 (D) $\dfrac{6}{5}$ (E) None of these

Solution:

(a)(12pts) Note that

$$y' = [\sec(x)]' = \sec(x)\tan(x)$$
$$\Rightarrow y'' = [\sec(x)\tan(x)]' = \sec(x)\tan(x) \cdot \tan(x) + \sec(x) \cdot \sec^2(x) \;\Rightarrow\; y'' = \sec(x)[\tan^2(x) + \sec^2(x)].$$

(b)(12pts) Since differentiability implies continuity, we first need $f(x)$ to be continuous at $x = 2$, that is, we need

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \;\Rightarrow\; -2 = 4a + b \;\Rightarrow\; b = -(2 + 4a)$$

and we need the derivative to exist at $x = 2$. For differentiability at $x = 2$, we need

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} \;\Leftrightarrow\; 2x - 3\Big|_{x=2} = 2ax\Big|_{x=2} \;\Rightarrow\; 1 = 4a \;\Rightarrow\; a = \frac{1}{4} \;\Rightarrow\; b = -(2 + 4a) = -3.$$

(c)(4pts) $\boxed{\text{Choice D.}}$ Note that

$$h'(x) = \left[\sqrt{4 + 3f(x)}\right]' = \frac{1}{2}(4 + 3f(x))^{-1/2} \cdot 3f'(x)$$
$$\Rightarrow h'(1) = \frac{1}{2}(4 + 3f(1))^{-1/2} \cdot 3f'(1) = \frac{1}{2}(4 + 3 \cdot 7)^{-1/2} \cdot 3 \cdot 4 = \frac{12}{2\sqrt{25}} = \frac{6}{5} \Rightarrow \text{Choice (D)}$$