1. (35pts) The following problems are not related.

(a)(16pts) Consider the following problem: *Ralphie flies a kite at a constant height of 150 ft above the ground with the wind carrying the kite horizontally away from Ralphie at a rate of 25 ft/sec. How fast must Ralphie let out the string when 250 ft of string is out?* **Now answer the following questions:** (i)(4pts) Write down the information that is given in this problem (use the notation established in the diagram below). (ii)(4pts) Write down what you are trying to find in this problem. (iii)(8pts) Solve the problem. Simplify your answer. (*Hint:* Recall that $3^2 + 4^2 = 5^2$)

(b)(16pts) Find the *linearizaton* $L(x)$ of $f(x) = \sqrt[3]{1+x}$ at $x = 0$ and use it to approximate $\sqrt[3]{0.95}$

(c)(3pts) Which choice below is the correct *derivative* of $f(x) = \dfrac{3x+2}{2x+3}$? (**No justification necessary** - *Choose only one answer, copy down the entire answer.*)

(A) $f'(x) = -\dfrac{1}{(2x+3)^2}$ (B) $f'(x) = \dfrac{3}{2}$ (C) $f'(x) = \dfrac{13}{(2x+3)^2}$ (D) $f'(x) = -\dfrac{5}{(2x+3)^2}$ (E) $f'(x) = \dfrac{5}{(2x+3)^2}$

Solution:

(a)(i)(4pts) We are $\boxed{\text{given } \dfrac{dx}{dt} = 25\,\text{ft/sec.}}$

(a)(ii)(4pts) We wish to $\boxed{\text{find } \dfrac{dz}{dt} \text{ when } z = 250\,\text{ft.}}$

(a)(iii)(8pts) The kite string, z, forms the hypotenuse of a right triangle. Thus

$$z^2 = x^2 + 150^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

and when $z = 250$ we have a "3-4-5" right triangle (as per the hint) and so $x = 4 \cdot 50 = 200$, that is,

$$250^2 = x^2 + 150^2 \Rightarrow (5 \cdot 50)^2 = x^2 + (3 \cdot 50)^2 \Rightarrow x^2 = (4 \cdot 50)^2 \Rightarrow x = 4 \cdot 50 = 200.$$

Thus

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = \frac{200}{250} \cdot 25 = 20\,\text{ft/sec}$$

so Ralphie must let the string out at $\boxed{20\,\text{ft/sec.}}$

(b)(16pts) Note that

$$f'(x) = \frac{1}{3}(1+x)^{-2/3} \cdot 1 \Rightarrow f'(0) = \frac{1}{3} \Rightarrow L(x) = f(0) + f'(0)x \Rightarrow \boxed{L(x) = 1 + \frac{1}{3}x}$$

thus

$$\sqrt[3]{0.95} = f(-0.05) \approx L(-0.05) = L\left(-\frac{1}{20}\right) = 1 - \frac{1}{3} \cdot \frac{1}{20} = \boxed{59/60}$$

(c)(3pts) $\boxed{\text{Choice E.}}$ Using the Quotient Rule, we have

$$\left[\frac{3x+2}{2x+3}\right]' = \frac{3 \cdot (2x+3) - 2 \cdot (3x+2)}{(2x+3)^2} = \frac{(6x+9) - (6x+4)}{(2x+3)^2} = \frac{5}{(2x+3)^2} \quad \text{thus} \quad \boxed{f'(x) = \frac{5}{(2x+3)^2}}$$

2. (34pts) The following problems are not related.

(a)(17pts) Suppose x represents the edge length of a metal cube. (i)(8pts) If $V(x) = x^3$ find dV, the *differential* of V. (ii)(9pts) Suppose the edge of the cube was originally found to be 10 cm in length but expands due to heat to 10.01 cm, use differentials to estimate the change in the volume, ΔV.

(b)(17pts) Find the *absolute minimum* and *absolute maximum values* of $f(x) = x^3 + 6x^2 + 1$ on the interval $[-1, 1]$. Give your answer in the form (x, y). Show all work, justify your answers and clearly label your answers.

Solution:

(a)(i)(8pts) If $V = x^3$ then $\boxed{dV = 3x^2 dx.}$

(a)(ii)(9pts) Note that $\Delta x = 10.01 - 10 = 0.01$ thus

$$\Delta V \approx dV = 3x^2 dx\Big|_{x=10, dx=\Delta x} = 3(10)^2(0.01) = 3 \text{ cm}^3$$

so the $\boxed{\text{volume will increase by approximately 3 cm}^3.}$

(b)(17pts) Here we have

$$f(x) = x^3 + 6x^2 + 1 \Rightarrow f'(x) = 3x^2 + 12x = 3x(x+4) \quad \text{and so} \quad f'(x) = 0 \Rightarrow x = 0, -4$$

but $x = -4$ is not in the interval $[-1, 1]$. Now plugging in the endpoints and critical points into $f(x)$ yields

$$f(-1) = -1 + 6 + 1 = 6,\ f(0) = 1,\ \text{ and } f(1) = 1 + 6 + 1 = 8$$

thus an $\boxed{\text{absolute maximum occurs at } (1, 8) \text{ and and absolute minimum occurs at } (0, 1).}$

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3. (34pts) The following problems are not related.

(a)(17pts) (i)(8pts) State the *Mean Value Theorem.* (ii)(9pts) Find all numbers c that satisfy the conclusion of the Mean Value Theorem for the function $g(x) = 1/x$ over the interval $[1, 3]$.

(b)(17pts) Suppose $y = f(x)$, use *implicit differentiation* to find y' if $\cos(xy) = 1 - \sin(y)$.

Solution:

(a)(i)(8pts) Mean Value Theorem: If $\boxed{f(x) \text{ is continuous on } [a, b]}$ and $\boxed{\text{differentiable on } (a, b)}$ then $\boxed{\text{there exists a number } c \text{ in the interval } (a, b)}$ such that $\boxed{f'(c) = \dfrac{f(b) - f(a)}{b - a}.}$

(a)(ii)(9pts) Note that $g(x)$ is continuous on $[1, 3]$ and differentiable on $(1, 3)$ and

$$g'(x) = -\frac{1}{x^2} \quad \text{and} \quad \frac{g(3) - g(1)}{3 - 1} = \frac{1/3 - 1}{2} = \frac{-2/3}{2} = -\frac{1}{3} \Rightarrow -\frac{1}{x^2} = -\frac{1}{3} \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

and so $\boxed{x = \sqrt{3} \text{ satisfies the Mean Value Theorem for } g(x) = 1/x \text{ over the interval } [1, 3].}$

(b)(17pts) Note that differentiating both sides of the equation $\cos(xy) = 1 - \sin(y)$ with respect to x yields

$$-\sin(xy) \cdot [y + xy'] = 0 - \cos(y)y' \Rightarrow \cos(y)y' - x\sin(xy)y' = y\sin(xy) \Rightarrow \boxed{y' = \frac{y\sin(xy)}{\cos(y) - x\sin(xy)}}$$

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4. (35pts) The following problems are not related.

(a)(16pts) Is the function $f(x) = \begin{cases} \dfrac{\sin(x)}{6x}, & \text{if } x < 0 \\ \dfrac{x^2+x+2}{6x^2+12}, & \text{if } x \geq 0 \end{cases}$ *continuous* at $x = 0$? Justify your answer with <u>limits</u>.

(b)(16pts) (*i*)(8pts) Write down the *piecewise* definition of the function $g(x) = |x^2 - 1|$.

(*ii*)(8pts) Find the derivative of $g(x) = |x^2 - 1|$.

(c)(3pts) The function $h(x) = \dfrac{3x+1}{\sqrt{4x^2+5}}$ has a *horizontal asymptote* at which choice below? (**No justification necessary** - *Choose only <u>one</u> answer, copy down the entire answer.*)

(A) $y=0$ (B) $y=\dfrac{3}{2}$ (C) $y=0$ and $y=3/2$ (D) $y=-3/2$ and $y=1.5$ (E) None of these

Solution: (a)(16pts) Note that using the "Special Limit" $\lim\limits_{\theta\to 0} \sin(\theta)/\theta = 1$ we have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(x)}{6x} = \lim_{x\to 0^-} \underbrace{\frac{\sin(x)}{x}}_{\text{goes to 1}} \cdot \frac{1}{6} = \frac{1}{6}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+x+2}{6x^2+12} = \frac{0+2}{0+12} = \frac{1}{6} = f(0)$$

thus we have shown

$$\boxed{\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \text{ thus } f(x) \text{ is continuous at } x = 0.}$$

(b)(*i*)(8pts) Note that

$$g(x) = |x^2-1| = |(x-1)(x+1)| = \begin{cases} x^2-1, & \text{if } x \leq -1 \text{ or } x \geq 1 \\ -(x^2-1), & \text{if } -1 < x < 1 \end{cases} = \boxed{\begin{cases} x^2-1, & \text{if } x \leq -1 \text{ or } x \geq 1 \\ 1-x^2, & \text{if } -1 < x < 1 \end{cases}}$$

(b)(*ii*)(8pts) Note that $g(x)$ is continuous at $x = -1$ and $x = 1$ but is not differentiable there, thus

$$\boxed{g'(x) = \begin{cases} 2x, & \text{if } x < -1 \text{ or } x > 1 \\ -2x, & \text{if } -1 < x < 1 \end{cases}}$$

(c)(3pts) $\boxed{\text{Choice D.}}$ Note that

$$\lim_{x\to\infty} \frac{3x+1}{\sqrt{4x^2+5}} = \lim_{x\to\infty} \frac{x(3+1/x)}{2|x|\sqrt{1+5/4x^2}} = \lim_{x\to\infty} \frac{x(3+1/x)}{2x\sqrt{1+5/4x^2}} = \lim_{x\to\infty} \frac{\cancel{x}(3+1/x)}{2\cancel{x}\sqrt{1+5/4x^2}} = \frac{3}{2} = 1.5$$

thus $y = 1.5$ is a horizontal asymptote of $h(x)$ and, similarly,

$$\lim_{x\to-\infty} \frac{3x+1}{\sqrt{4x^2+5}} = \lim_{x\to-\infty} \frac{x(3+1/x)}{2|x|\sqrt{1+5/4x^2}} = \lim_{x\to-\infty} \frac{x(3+1/x)}{2(-x)\sqrt{1+5/4x^2}} = \lim_{x\to-\infty} \frac{\cancel{x}(3+1/x)}{2(-\cancel{x})\sqrt{1+5/4x^2}} = -\frac{3}{2}$$

thus $y = -3/2$ is also a horizontal asymptote of $h(x)$.

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5. (12pts) Answer either **ALWAYS TRUE** or **FALSE**. You do <u>NOT</u> need to justify your answer. (*Don't just write down "A.T." or "F", completely write out the words "ALWAYS TRUE" or "FALSE" depending on your answer.*)

(a)(3pts) If $f(x)$ is *continuous* at $x = a$ then $f(x)$ is *differentiable* at $x = a$.

(b)(3pts) $\lim_{h \to 0} \dfrac{\sec(x+h) - \sec(x)}{h} = \sec(x)\tan(x)$.

(c)(3pts) Suppose the position function of a particle (at time $t \geq 0$ in seconds) is given by $s(t) = t^2 - t$ meters, then the *total distance* traveled during the time period $0 \leq t \leq 1$ by the particle is 0.25 meters.

(d)(3pts) If $f(x) = \sqrt{2x+3}$ and $g(x) = x^2 + 5$, then $(f \circ g)(x) = \sqrt{2x^2 + 13}$ for all real numbers x.

Solution: 3pts each: (a) FALSE (b) ALWAYS TRUE (c) FALSE (d) ALWAYS TRUE

Discussion:

(a) False. A good counterexample of this is $f(x) = |x|$ which is continuous at $x = 0$ but is clearly not differentiable at $x = 0$.

(b) Always True. If we let $f(x) = \sec(x)$ then

$$\lim_{h \to 0} \frac{\sec(x+h) - \sec(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \sec(x)\tan(x)$$

(c) False. The total distance travelled is 1/2. Note that $v(t) = s'(t) = 2t - 1$ and $v'(t) = 0$ when $t = 1/2$ thus

$$\text{Total Distance} = |s(0) - s(1/2)| + |s(1) - s(1/2)| = |0 - (-1/4)| + |0 - (-1/4)| = 1/2$$

(d) Always True. Note that $g(x) = x^2 + 5$ is defined for all real numbers thus

$$(f \circ g)(x) = f(x^2 + 5) = \sqrt{2(x^2+5)+3} = \sqrt{2x^2 + 10 + 3} = \sqrt{2x^2 + 13}$$

with all real numbers as the domain of $(f \circ g)(x)$.