1. (24pts) The following problems are not related. Show all work.

(a)(12pts) Use the *Quotient Rule* to find $f'(x)$ if $f(x) = \dfrac{x^3}{1-x^2}$. Simplify your answer.

(b)(12pts) If $y = \sec^4(\pi\theta)$ find $dy/d\theta$. Simplify your answer.

Solution:

(a)(12pts) Note

$$\left[\frac{x^3}{1-x^2}\right]' = \frac{3x^2(1-x^2) - x^3(-2x)}{(1-x^2)^2} = \frac{x^2[3-3x^2+2x^2]}{(1-x^2)^2} = \boxed{\frac{x^2[3-x^2]}{(1-x^2)^2}}.$$

(b)(12pts) By the Chain Rule, we have

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[(\sec(\pi\theta))^4\right] = 4\sec^3(\pi\theta)\cdot\sec(\pi\theta)\tan(\pi\theta)\cdot\pi = \boxed{4\pi\sec^4(\pi\theta)\tan(\pi\theta).}$$

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2. (28pts) The following problems are not related. Show all work.

(a)(12pts) Suppose $y = f(x)$, use *implicit differentiation* to find y' if $\cos(xy) = 1 - \sin(y)$.

(b)(12pts) Find the equation of the *tangent line* to $y = x^2 - x^4$ at the point $(1, 0)$. Simplify your answer.

(c)(4pts) Which of the choices below is equivalent to the limit $\displaystyle\lim_{x\to\pi/3}\frac{\cos(x)-0.5}{x-{}^{\pi}/3}$? **Choose only <u>one</u> answer.** *No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.*

(A) $\dfrac{0}{0}$ (B) $+\infty$ (C) $\dfrac{1}{2}$ (D) $-\dfrac{\sqrt{3}}{2}$ (E) None of these

Solution:

(a)(12pts) Note that differentiating both sides of the equation $\cos(xy) = 1 - \sin(y)$ with respect to x yields

$$-\sin(xy)\cdot[y + xy'] = 0 - \cos(y)y' \Rightarrow \cos(y)y' - x\sin(xy)y' = y\sin(xy) \Rightarrow \boxed{y' = \frac{y\sin(xy)}{\cos(y) - x\sin(xy)}}$$

(b)(12pts) Note that here $(a, f(a)) = (1, 0)$ and so the equation of the tangent line is $y - 0 = f'(1)(x-1)$ where

$$f'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \left.\left(2x - 4x^3\right)\right|_{x=1} = 2 - 4 = -2 \Rightarrow y = -2(x-1) \Rightarrow \boxed{y = -2x + 2}$$

(c)(4pts) $\boxed{\text{Choice D.}}$ Note that if we let $f(x) = \cos(x)$ then, using the limit definition of the derivative, we have

$$\lim_{x\to\pi/3}\frac{\cos(x)-0.5}{x-{}^{\pi}/3} = \lim_{x\to\pi/3}\frac{f(x)-f({}^{\pi}/3)}{x-{}^{\pi}/3} = f'({}^{\pi}/3) = -\sin(x)\Big|_{x={}^{\pi}/3} = -\sin({}^{\pi}/3) = -\frac{\sqrt{3}}{2} \Rightarrow \text{D}$$

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3. (20pts) The following problems are not related. Show all work.

(a)(10pts) For what value(s) of $x \in [0, 2\pi]$ does $f(x) = x + 2\sin(x)$ have *horizontal tangent*?

(b) The position function of a particle is given by $s(t) = t^3 - 6t^2 + 9t$ where $t \geq 0$ is in seconds and distance is in feet. (i)(5pts) When is the particle at rest? (ii)(5pts) What is the *total distance* traveled by the particle in the first 2 seconds? Simplify your answer.

Solution: (a)(10pts) Note that

$$f'(x) = 0 \Rightarrow 1 + 2\cos(x) = 0 \Rightarrow \cos(x) = -\frac{1}{2} \Rightarrow \boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3}.}$$

(b)(i)(5pts) Here we have

$$v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3) \Rightarrow \boxed{\text{particle at rest when } t = 1, 3.}$$

(b)(ii)(5pts) The total distance in the first 2 seconds is

$$|s(1) - s(0)| + |s(1) - s(2)| = |4 - 0| + |4 - 2| = 6 \text{ ft} \Rightarrow \boxed{\text{total distance travelled is 6ft.}}$$

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4. (28pts) The following problems are not related. Show all work.

(a)(12pts) Suppose $\cot(g'(x)) = x\pi^2 + 1$, find $g''(0)$.

(b)(12pts) If $f(x) = \begin{cases} \frac{x}{2}, & \text{if } x < 2 \\ \sqrt{2x}, & \text{if } x \geq 2 \end{cases}$, find $f'(x)$. Is $f(x)$ *differentiable* for all x? Explain.

(c)(4pts) Suppose the equation of the tangent line to the function $f(x) = ax^2 + bx$ at $(1, 1)$ is $y = 3x - 2$, then a and b are equal to which choice below? **Choose only <u>one</u> answer.** *No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.*

(A) $a = 1, b = 1$ (B) $a = 2, b = -1$ (C) $a = 3, b = 4$ (D) $a = -3, b = 4$ (E) None of these

Solution:

(a)(12pts) Differentiation both sides of $\cot(g'(x)) = x\pi^2 + 1$ with respect to x yields

$$-\csc^2(g'(x)) \cdot g''(x) = \pi^2 \Rightarrow g''(x) = \frac{-\pi^2}{\csc^2(g'(x))} \Rightarrow \boxed{g''(0) = \frac{-\pi^2}{\csc^2(g'(0))}}$$

(b)(12pts) First note that $f(x)$ is <u>not</u> continuous at $x = 2$ therefore

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x < 2 \\ \sqrt{2x}, & \text{if } x \geq 2 \end{cases} \Longrightarrow \boxed{f'(x) = \begin{cases} \frac{1}{2}, & \text{if } x < 2 \\ \frac{1}{\sqrt{2x}}, & \text{if } x > 2 \end{cases}}$$

No, since $f(x)$ is not continuous at $x = 2$ this implies $\boxed{f(x) \text{ is } \underline{\text{not}} \text{ differentiable at } x = 2.}$

(c)(4pts) Choice B. Note that $f(x) = ax^2 + bx$ and $f(1) = 1$ implies $1 = a + b$ and also note that $f'(1)$ is equal to the slope of the tangent line at $(1, 1)$ which is $y = 3x - 2$ so $f'(1) = 3$ which implies $(2ax + b)\big|_{x=1} = 3$, that is, $2a + b = 3$ and $a + b = 1$ thus $a = 2, b = -1 \Rightarrow$ B.
