1. (24pts) The following problems are not related. Show all work.

(a)(12pts) Use the Quotient Rule to find \( f'(x) \) if \( f(x) = \frac{x^3}{1-x^2} \). Simplify your answer.

(b)(12pts) If \( y = \sec^4(\pi \theta) \) find \( \frac{dy}{d\theta} \). Simplify your answer.

Solution:

(a)(12pts) Note
\[
\left( \frac{x^3}{1-x^2} \right)' = \frac{3x^2(1-x^2) - x^3(-2x)}{(1-x^2)^2} = \frac{x^3(3-3x^2+2x^2)}{(1-x^2)^2} = \frac{x^3(3-x^2)}{(1-x^2)^2}.
\]

(b)(12pts) By the Chain Rule, we have
\[
\frac{dy}{d\theta} = \frac{d}{d\theta} (\sec(\pi \theta))^4 = 4 \sec^3(\pi \theta) \cdot \sec(\pi \theta) \tan(\pi \theta) \cdot \pi = 4\pi \sec^4(\pi \theta) \tan(\pi \theta).
\]

2. (28pts) The following problems are not related. Show all work.

(a)(12pts) Suppose \( y = f(x) \), use implicit differentiation to find \( y' \) if \( \cos(xy) = 1 - \sin(y) \).

(b)(12pts) Find the equation of the tangent line to \( y = x^2 - x^4 \) at the point \((1,0)\). Simplify your answer.

(c)(4pts) Which of the choices below is equivalent to the limit \( \lim_{x \to \pi/3} \frac{\cos(x) - 0.5}{x - \pi/3} \)? Choose only one answer. No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.

(A) 0
(B) \( \infty \)
(C) 1/2
(D) \( -\frac{\sqrt{3}}{2} \)
(E) None of these

Solution:

(a)(12pts) Note that differentiating both sides of the equation \( \cos(xy) = 1 - \sin(y) \) with respect to \( x \) yields
\[
-\sin(xy) \cdot [y + xy'] = 0 - \cos(y)y' \Rightarrow \cos(y)y' - x\sin(xy)y' = y\sin(xy) \Rightarrow y' = \frac{y\sin(xy)}{\cos(y) - x\sin(xy)}
\]

(b)(12pts) Note that here \((a, f(a)) = (1,0)\) and so the equation of the tangent line is \( y - 0 = f'(1)(x - 1) \) where
\[
f'(1) = \frac{dy}{dx} \bigg|_{x=1} = (2x - 4x^3) \bigg|_{x=1} = 2 - 4 = -2 \Rightarrow y = -2(x - 1) \Rightarrow y = -2x + 2
\]

(c)(4pts) Choice D. Note that if we let \( f(x) = \cos(x) \) then, using the limit definition of the derivative, we have
\[
\lim_{x \to \pi/3} \frac{\cos(x) - 0.5}{x - \pi/3} = \lim_{x \to \pi/3} \frac{f(x) - f(\pi/3)}{x - \pi/3} = f'(\pi/3) = -\sin(x) \bigg|_{x=\pi/3} = -\sin(\pi/3) = -\frac{\sqrt{3}}{2} \Rightarrow D
\]
3. (20pts) The following problems are not related. Show all work.

(a)(10pts) For what value(s) of \( x \in [0, 2\pi] \) does \( f(x) = x + 2 \sin(x) \) have horizontal tangent?

Solution: (a)(10pts) Note that

\[
f'(x) = 1 + 2 \cos(x) = 0 \Rightarrow \cos(x) = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}.
\]

(b) The position function of a particle is given by \( s(t) = t^3 - 6t^2 + 9t \) where \( t \geq 0 \) is in seconds and distance is in feet. (i)(5pts) When is the particle at rest? (ii)(5pts) What is the total distance traveled by the particle in the first 2 seconds? Simplify your answer.

Solution: (b)(i)(5pts) Here we have

\[
v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3) \Rightarrow \text{particle at rest when } t = 1, 3.
\]

(b)(ii)(5pts) The total distance in the first 2 seconds is

\[
|s(1) - s(0)| + |s(1) - s(2)| = |4 - 0| + |4 - 2| = 6 \text{ ft} \Rightarrow \text{total distance travelled is 6 ft.}
\]

4. (28pts) The following problems are not related. Show all work.

(a)(12pts) Suppose \( \cot(g'(x)) = x\pi^2 + 1 \), find \( g''(0) \).

(b)(12pts) If \( f(x) = \begin{cases} \frac{x}{2}, & \text{if } x < 2 \\ \sqrt{2x}, & \text{if } x \geq 2 \end{cases} \), find \( f'(x) \). Is \( f(x) \) differentiable for all \( x \)? Explain.

(c)(4pts) Suppose the equation of the tangent line to the function \( f(x) = ax^2 + bx \) at \((1, 1)\) is \( y = 3x - 2 \), then \( a \) and \( b \) are equal to which choice below? **Choose only one answer.** No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.

(A) \( a = 1, b = 1 \)  
(B) \( a = 2, b = -1 \)  
(C) \( a = 3, b = 4 \)  
(D) \( a = -3, b = 4 \)  
(E) None of these

Solution:

(a)(12pts) Differentiation both sides of \( \cot(g'(x)) = x\pi^2 + 1 \) with respect to \( x \) yields

\[
-csc^2(g'(x)) \cdot g''(x) = \pi^2 \Rightarrow g''(x) = \frac{-\pi^2}{csc^2(g'(x))} \Rightarrow g''(0) = \frac{-\pi^2}{csc^2(g'(0))}
\]

(b)(12pts) First note that \( f(x) \) is not continuous at \( x = 2 \) therefore

\[
f(x) = \begin{cases} \frac{x}{2}, & \text{if } x < 2 \\ \sqrt{2x}, & \text{if } x \geq 2 \end{cases} \Rightarrow f'(x) = \begin{cases} \frac{1}{2}, & \text{if } x < 2 \\ \frac{1}{\sqrt{2x}}, & \text{if } x > 2 \end{cases}
\]

No, since \( f(x) \) is not continuous at \( x = 2 \) this implies \( f(x) \) is not differentiable at \( x = 2 \).
(c)(4pts) **Choice B.** Note that \( f(x) = ax^2 + bx \) and \( f(1) = 1 \) implies \( 1 = a + b \) and also note that \( f'(1) \) is equal to the slope of the tangent line at \((1, 1)\) which is \( y = 3x - 2 \) so \( f'(1) = 3 \) which implies \((2ax + b)|_{x=1} = 3\), that is, \( 2a + b = 3 \) and \( a + b = 1 \) thus \( a = 2, b = -1 \) \( \Rightarrow \) B.