Solution: APPM 1340 Exam 2 (Solutions) Fall 2019

1. (28pts) The following problems are not related.
   (a)(12pts) Suppose \( f(x) = \sin(x) \), with domain \( 0 \leq x \leq 4\pi \) and \( g(x) = x^{1/2} \). Find \((g \circ f)(x)\) and also state the domain in interval notation. Justify your answer.
   
   (b)(12pts) Suppose that \( h(x) = \begin{cases} \frac{|x - 3|}{x^2 - 9} & \text{if } x < 3 \\ \sqrt[3]{3x - 5} & \text{if } x > 3 \end{cases} \), find the two-sided limit \( \lim_{x \to 3} h(x) \). Show all work and justify your answer.
   
   (c)(4pts) The function \( f(x) = \frac{3x + 1}{\sqrt{4x^2 + 5}} \) has a horizontal asymptote at which choice below? (No justification necessary - Choose only one answer, copy down the entire answer.)
   
   (A) \( y = 0 \)  \hspace{1cm} (B) \( y = \frac{3}{2} \)  \hspace{1cm} (C) \( y = 0 \) and \( y = \frac{3}{2} \)  \hspace{1cm} (D) \( y = -\frac{3}{2} \) and \( y = 1.5 \)  \hspace{1cm} (E) None of these
   
   Solution: (a)(12pts) Note that
   \[
   (g \circ f)(x) = g(f(x)) = g(\sin(x)) = \sqrt{\sin(x)} \quad \text{with domain } [0, \pi] \cup [2\pi, 3\pi] \cup [4\pi, 4\pi].
   \]

   so \( (g \circ f)(x) = \sqrt{\sin(x)} \), with domain \([0, \pi] \cup [2\pi, 3\pi] \cup [4\pi, 4\pi]\).
   
   (b)(12pts) We need to examine the 1-sided limits:
   \[
   \lim_{x \to 3^-} h(x) = \lim_{x \to 3^-} \frac{|x - 3|}{x^2 - 9} = \lim_{x \to 3^-} \frac{(x - 3)}{x^2 - 9} = \lim_{x \to 3^-} \frac{-(x - 3)}{(x - 3)(x + 3)} = \lim_{x \to 3^-} \frac{-1}{x + 3} = -1/6
   \]

   and
   \[
   \lim_{x \to 3^+} h(x) = \lim_{x \to 3^+} \frac{\sqrt{3x - 5}}{12} = \frac{\sqrt{9} - 5}{12} = \frac{-2}{12} = -\frac{1}{6}
   \]

   thus \( \lim_{x \to 3} h(x) = -1/6 \).
   
   (c)(4pts) Choice D. Note that
   \[
   \lim_{x \to \infty} \frac{3x + 1}{\sqrt{4x^2 + 5}} = \lim_{x \to \infty} \frac{x(3 + 1/x)}{2|x| \sqrt{1 + 5/4x^2}} = \lim_{x \to \infty} \frac{x(3 + 1/x)}{2x \sqrt{1 + 5/4x^2}} = \lim_{x \to \infty} \frac{x(3 + 1/x)}{2x \sqrt{1 + 5/4x^2}} = \frac{3}{2} = 1.5
   \]

   thus \( y = 1.5 \) is a horizontal asymptote of \( f(x) \) and, similarly,
   \[
   \lim_{x \to -\infty} \frac{3x + 1}{\sqrt{4x^2 + 5}} = \lim_{x \to -\infty} \frac{x(3 + 1/x)}{2|x| \sqrt{1 + 5/4x^2}} = \lim_{x \to -\infty} \frac{x(3 + 1/x)}{2x \sqrt{1 + 5/4x^2}} = \lim_{x \to -\infty} \frac{x(3 + 1/x)}{2x \sqrt{1 + 5/4x^2}} = \frac{3}{2} = -3/2
   \]

   thus \( y = -3/2 \) is also a horizontal asymptote of \( f(x) \).

2. (24pts) The following problems are not related.
   
   (a)(12pts) Use the Squeeze Theorem to evaluate the following limit: \( \lim_{x \to 1} (x - 1)^2 \cos \left( \frac{1}{x - 1} \right) \). Show all work, explain your answer.
   
   (b)(12pts) Find the limit \( \lim_{x \to 0} \frac{\sin(\pi x)}{\sin(5x)} \). Justify your answer, show all work.
3. (28pts) The following problems are not related.

(a) (12pts) Evaluate the limit: \( \lim_{x \to 4} \frac{\sqrt{6x + 1} - 5}{x - 4} \). Show all work.

(b) (12pts) Suppose \( f(x) = \begin{cases} x^2 + x, & \text{if } x \neq 0 \\ \cos(x), & \text{if } x = 0 \end{cases} \). (i) (6pts) Find the \( \lim_{x \to 0} f(x) \). (ii) (6pts) Is \( f(x) \) continuous for all real \( x \)? If not, classify the discontinuities of \( f(x) \). Use limits to answer this question. Explain.

(c) (4pts) Which choice below would result in shifting the graph of \( y = s(t) \) one unit to the right and then reflecting it about the \( y \)-axis? (No justification necessary - Choose only one answer, copy down the entire answer.)

(A) \( y = -s(t) - 1 \) \hspace{2em} (B) \( y = s(-(t + 1)) \) \hspace{2em} (C) \( y = s(-(t - 1)) \) \hspace{2em} (D) \( y = -s(t + 1) \) \hspace{2em} (E) \( y = s(-t) - 1 \)

Solution: (a) (12pts) Multiplying by the conjugate yields:

\[
\lim_{x \to 4} \frac{\sqrt{6x + 1} - 5}{x - 4} = \lim_{x \to 4} \frac{\sqrt{6x + 1} - 5}{x - 4} \cdot \frac{\sqrt{6x + 1} + 5}{\sqrt{6x + 1} + 5} = \lim_{x \to 4} \frac{(6x + 1) - 25}{(x - 4)(\sqrt{6x + 1} + 5)} = \lim_{x \to 4} \frac{6(x - 4)}{(x - 4)(\sqrt{6x + 1} + 5)} = \frac{6}{10} = \frac{3}{5}.
\]

(b)(i) (6pts) Note that

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0^+} x^2 + x = 0 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + x = 0 \quad \text{thus} \quad \lim_{x \to 0} f(x) = 0
\]

(b)(ii) (6pts) No, \( f(x) \) is not continuous at \( x = 0 \). For \( x \neq 0 \), we have \( f(x) = x^2 + x \), and recall that polynomial are continuous and at \( x = 0 \) we have to check that \( \lim_{x \to 0} f(x) = f(0) \), note that

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 + x = 0^2 + 0 = 0 \quad \text{and} \quad f(0) = \cos(0) = 1 \Rightarrow \lim_{x \to 0} f(x) \neq f(0)
\]

and so \( f(x) \) is not continuous for all real \( x \) and has a removable discontinuity at \( x = 0 \).

(c)(4pts) Choice B. Note that shifting the graph of \( y = s(t) \) one unit to the right and then reflecting the graph about the \( y \)-axis is the same as reflecting the graph of \( s(t) \) about the \( y \)-axis, i.e. \( s(-t) \), and then shifting the graph one unit to the left, thus we have \( s(-(t + 1)) \). (Note Choice C is incorrect since the transformation \( y = s(-(t - 1)) \) shifts the graph on unit to the right and then reflects the graph about the vertical line \( t = 1 \) not the \( y \)-axis.)
4. (20pts) The following problems are not related.

(a) (12pts) Let \( q(t) = \begin{cases} \frac{kt^2 + 2}{t^2 - 9}, & \text{if } t \leq 3 \\ \frac{t^2 - 9}{t - 3}, & \text{if } t > 3 \end{cases} \). Find the value of \( k \) that makes \( q(t) \) continuous at \( t = 3 \). Justify.

(b) (8pts) Suppose the function \( y = g(x) \) has horizontal asymptote \( y = 3 \) and vertical asymptote \( x = -1 \), find all horizontal and vertical asymptotes of the function \( h(t) = -g(t - 2)/3 \). Justify your answer.

**Solution:**

(a) (12pts) We check the one-sided limits, thus

\[
q(3) = \lim_{t \to 3^-} q(t) = \lim_{t \to 3^-} \frac{kt^2 + 2}{t^2 - 9} = \frac{9k + 2}{6} \quad \text{and} \quad \lim_{t \to 3^+} q(t) = \lim_{t \to 3^+} \frac{t^2 - 9}{t - 3} = 6
\]

and letting \( 9k + 2 = 6 \) we see that \( k = 4/9 \) and so we see that \( q(t) \) will be continuous at \( t = 3 \) if \( k = 4/9 \).

(b) (8pts) Note that we can assume that \( \lim_{x \to \infty} g(x) = 3 \) and \( \lim_{x \to -\infty} g(x) = 3 \) and so

\[
\lim_{t \to \infty} h(t) = \lim_{t \to \infty} -g(t - 2)/3 = -\frac{1}{3} \lim_{t \to \infty} g(t - 2) = -\frac{1}{3} \cdot 3 = -1
\]

and, similarly, \( \lim_{t \to -\infty} h(t) = -1 \) and so \( y = -1 \) is a horizontal asymptote of \( h(t) \). Now note that the graph of \( h(t) = -g(t - 2)/3 \) is the graph of \( g(t) \) reflected about the \( t \)-axis, re-scaled and shifted to the right 2 units. Thus \( h(t) \) has a vertical asymptote at \( t = -1 + 2 = 1 \). So \( h(t) \) has a horizontal asymptote at \( y = -1 \) and a vertical asymptote at \( t = 1 \).