1. (28pts) The following problems are not related.

(a)(12pts) Suppose $f(x) = \sin(x)$, with domain $0 \le x \le 4\pi$ and $g(x) = x^{1/2}$. Find $(g \circ f)(x)$ and also state the domain in interval notation. Justify your answer.

(b)(12pts) Suppose that $h(x) = \begin{cases} \dfrac{|x-3|}{x^2-9}, & \text{if } x < 3 \\ \dfrac{\sqrt{3x}-5}{12}, & \text{if } x > 3 \end{cases}$, find the two-sided limit $\lim_{x\to 3} h(x)$. Show all work and justify your answer.

(c)(4pts) The function $f(x) = \dfrac{3x+1}{\sqrt{4x^2+5}}$ has a *horizontal asymptote* at which choice below? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $y=0$ (B) $y=\dfrac{3}{2}$ (C) $y=0$ and $y=3/2$ (D) $y=-3/2$ and $y=1.5$ (E) None of these

Solution: (a)(12pts) Note that

$$(g \circ f)(x) = g(f(x)) = \underbrace{g(\sin(x))}_{\text{need } 0\le x\le 4\pi} = \underbrace{\sqrt{\sin(x)}}_{\text{need } \sin(x)\ge 0} \text{ with domain } [0,\pi]\cup[2\pi,3\pi]\cup[4\pi,4\pi].$$

so $\boxed{(g \circ f)(x) = \sqrt{\sin(x)}, \text{ with domain } [0,\pi]\cup[2\pi,3\pi]\cup[4\pi,4\pi].}$

(b)(12pts) We need to examine the 1-sided limits:

$$\lim_{x\to 3^-} h(x) = \lim_{x\to 3^-} \frac{|x-3|}{x^2-9} = \lim_{x\to 3^-} \frac{-(x-3)}{x^2-9} = \lim_{x\to 3^-} \frac{-(x-3)}{(x-3)(x+3)} = \lim_{x\to 3^-} \frac{-1}{(x+3)} = -1/6$$

and

$$\lim_{x\to 3^+} h(x) = \lim_{x\to 3^+} \frac{\sqrt{3x}-5}{12} = \frac{\sqrt{9}-5}{12} = \frac{-2}{12} = -\frac{1}{6}$$

thus $\boxed{\lim_{x\to 3} h(x) = -1/6.}$

(c)(4pts) $\boxed{\text{Choice D.}}$ Note that

$$\lim_{x\to\infty} \frac{3x+1}{\sqrt{4x^2+5}} = \lim_{x\to\infty} \frac{x(3+1/x)}{2|x|\sqrt{1+5/4x^2}} = \lim_{x\to\infty} \frac{x(3+1/x)}{2x\sqrt{1+5/4x^2}} = \lim_{x\to\infty} \frac{\cancel{x}(3+1/x)}{2\cancel{x}\sqrt{1+5/4x^2}} = \frac{3}{2} = 1.5$$

thus $y = 1.5$ is a horizontal asymptote of $f(x)$ and, similarly,

$$\lim_{x\to -\infty} \frac{3x+1}{\sqrt{4x^2+5}} = \lim_{x\to -\infty} \frac{x(3+1/x)}{2|x|\sqrt{1+5/4x^2}} = \lim_{x\to -\infty} \frac{x(3+1/x)}{2(-x)\sqrt{1+5/4x^2}} = \lim_{x\to -\infty} \frac{\cancel{x}(3+1/x)}{2(-\cancel{x})\sqrt{1+5/4x^2}} = -\frac{3}{2}$$

thus $y = -3/2$ is also a horizontal asymptote of $f(x)$.

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2. (24pts) The following problems are not related.

(a)(12pts) Use the Squeeze Theorem to evaluate the following limit: $\lim_{x\to 1} (x-1)^2 \cos\left(\dfrac{1}{x-1}\right)$. Show all work, explain your answer.

(b)(12pts) Find the limit $\lim_{x\to 0} \dfrac{\sin(\pi x)}{\sin(5x)}$. Justify your answer, show all work.

Solution: (a)(12pts) Note that

$$-1 \leq \cos\left(\frac{1}{x-1}\right) \leq 1 \Longrightarrow -(x-1)^2 \leq (x-1)^2\cos\left(\frac{1}{x-1}\right) \leq (x-1)^2$$

and since $\lim_{x\to 1} -(x-1)^2 = \lim_{x\to 1}(x-1)^2 = 0$ thus, by the Squeeze Theorem, $\boxed{\lim_{x\to 1}(x-1)^2\cos\left(\frac{1}{x-1}\right) = 0.}$

(b)(12pts) Using the special limit $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$ we have

$$\lim_{x\to 0}\frac{\sin(\pi x)}{\sin(5x)} = \lim_{x\to 0}\frac{\sin(\pi x)}{\pi x}\cdot\frac{5x}{\sin(5x)}\cdot\frac{\pi}{5} = 1\cdot 1\cdot\frac{\pi}{5} = \boxed{\pi/5}$$

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3. (28pts) The following problems are not related.

(a)(12pts) Evaluate the limit: $\lim_{x\to 4}\frac{\sqrt{6x+1}-5}{x-4}$. Show all work.

(b)(12pts) Suppose $f(x) = \begin{cases} x^2+x, & \text{if } x \neq 0 \\ \cos(x), & \text{if } x = 0 \end{cases}$. (i)(6pts) Find the $\lim_{x\to 0} f(x)$. (ii)(6pts) Is $f(x)$ continuous for all real x? If not, classify the discontinuities of $f(x)$. Use limits to answer this question. Explain.

(c)(4pts) Which choice below would result in shifting the graph of $y = s(t)$ one unit to the right and then reflecting it about the y-axis? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $y = -s(t) - 1$ (B) $y = s(-(t+1))$ (C) $y = s(-(t-1))$ (D) $y = -s(t+1)$ (E) $y = s(-t) - 1$

Solution: (a)(12pts) Multiplying by the conjugate yields:

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{6x+1}-5}{x-4} &\overset{\text{"0/0"}}{=} \lim_{x\to 4}\frac{\sqrt{6x+1}-5}{x-4}\cdot\frac{\sqrt{6x+1}+5}{\sqrt{6x+1}+5} \\ &= \lim_{x\to 4}\frac{(6x+1)-25}{(x-4)[\sqrt{6x+1}+5]} = \lim_{x\to 4}\frac{6\cancel{(x-4)}}{\cancel{(x-4)}[\sqrt{6x+1}+5]} = 6/10 = \boxed{3/5.}\end{aligned}$$

(b)(i)(6pts) Note that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2+x = 0 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2+x = 0 \text{ thus } \boxed{\lim_{x\to 0} f(x) = 0}$$

(b)(ii)(6pts) No, $f(x)$ is not continuous at $x = 0$. For $x \neq 0$, we have $f(x) = x^2 + x$, and recall that polynomial are continuous and at $x = 0$ we have to check that $\lim_{x\to 0} f(x) = f(0)$, note that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 + x = 0^2 + 0 = 0 \text{ and } f(0) = \cos(0) = 1 \Rightarrow \lim_{x\to 0} f(x) \neq f(0)$$

and so $\boxed{f(x) \text{ is not continuous for all real } x \text{ and has a \textit{removable discontinuity} at } x = 0.}$

(c)(4pts) $\boxed{\text{Choice B.}}$ Note that shifting the graph of $y = s(t)$ one unit to the right and then reflecting the graph about the y-axis is the same as reflecting the graph of $s(t)$ about the y-axis, *i.e.* $s(-t)$, and then shifting the graph one unit to the left, thus we have $s(-(t+1))$. (Note Choice C is incorrect since the transformation $y = s(-(t-1))$ shifts the graph on unit to the right and then reflects the graph about the vertical line $t = 1$ not the y-axis.)

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4. (20pts) The following problems are not related.

(a)(12pts) Let $q(t) = \begin{cases} kt^2 + 2, & \text{if } t \le 3 \\ \dfrac{t^2 - 9}{t - 3}, & \text{if } t > 3 \end{cases}$. Find the value of k that makes $q(t)$ *continuous* at $t = 3$. Justify.

(b)(8pts) Suppose the function $y = g(x)$ has horizontal asymptote $y = 3$ and vertical asymptote $x = -1$, find all horizontal and vertical asymptotes of the function $h(t) = -g(t-2)/3$. Justify your answer.

Solution: (a)(12pts) We check the one-sided limits, thus

$$q(3) = \lim_{t\to 3^-} q(t) = \lim_{t\to 3^-} kt^2 + 2 = 9k + 2 \text{ and } \lim_{t\to 3^+} q(t) = \lim_{t\to 3^+} \frac{t^2 - 9}{t - 3} \overset{\text{“0/0”}}{=} \lim_{t\to 3^+} \frac{\cancel{(t-3)}(t+3)}{\cancel{(t-3)}} = 6$$

and letting $9k + 2 = 6$ we see that $k = 4/9$ and so we see that $\boxed{q(t) \text{ will be continuous at } t = 3 \text{ if } k = 4/9.}$

(b)(8pts) Note that we can assume that $\lim_{x\to\infty} g(x) = 3$ and $\lim_{x\to -\infty} g(x) = 3$ and so

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} -g(t-2)/3 = -\frac{1}{3} \lim_{t\to\infty} g(t-2) = -\frac{1}{3} \cdot 3 = -1$$

and, similarly, $\lim_{t\to -\infty} h(t) = -1$ and so $y = -1$ is a horizontal asymptote of $h(t)$. Now note that the graph of $h(t) = -g(t-2)/3$ is the graph of $g(t)$ reflected about the t-axis, re-scaled and shifted to the right 2 units. Thus $h(t)$ has a vertical asymptote at $t = -1+2 = 1$. So $h(t)$ has a $\boxed{\text{horizontal asymptote at } y = -1}$ and a $\boxed{\text{vertical asymptote at } t = 1.}$
