INSTRUCTIONS: Outside paper and electronic devices are **not** permitted. Exam is worth 150 points. Neatness counts. Unless indicated, answers with no supporting work may receive no credit. Simplify all your answers, and $\boxed{\text{BOX}}$ your final answers.

1. (20 points) Consider the function $f(x) = \sqrt{1-x}$.

   (a) Find the linearization of $f(x)$ at $a = 0$.

   (b) Use the linearization from part (a) to approximate $\sqrt{0.9}$. Answer to the nearest $\frac{1}{100}$th.

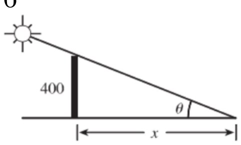
   Solution:

   (a) point $(0, 1)$, $f'(x) = \dfrac{1}{2}(1-x)^{-1} = \dfrac{-1}{2\sqrt{1-x}} \implies$ slope is $f'(0) = \dfrac{-1}{2}$

   $y - 1 = -\dfrac{1}{2}(x - 0)$ or $\boxed{y(x) = \dfrac{-x+2}{2}}$

   (b) $\sqrt{0.9} = f(0.1) \approx y(0.1) = \dfrac{1.9}{2} = \boxed{0.95}$

2. (14 points) In the diagram below, the angle of elevation of the sun, θ, is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building changing when the angle of elevation of the sun is $\dfrac{\pi}{6}$? Include units in your answer.



   The variable x represents the length of the buildings shadow.

   Solution: Given $\dfrac{d\theta}{dt} = -\dfrac{1}{4}$, find $\dfrac{dx}{dt}$ evaluated at $\theta = \dfrac{\pi}{6}$
   Note, $\theta = \dfrac{\pi}{6} \implies x = 400\sqrt{3}$ and $h = 800$.
   There are two approaches to this question.
   First:
   $\cot(\theta) = \dfrac{x}{400} \implies x = 400\cot(\theta) \implies \dfrac{dx}{dt} = -400\csc^2(\theta)\dfrac{d\theta}{dt} =$
   $-400 \cdot 4 \cdot (-\frac{1}{4}) = \boxed{400}$

   Second:
   $\tan(\theta) = \dfrac{400}{x} \implies \sec^2(\theta) \cdot \dfrac{d\theta}{dt} = -\dfrac{400}{x^2} \cdot \dfrac{dx}{dt} \implies \dfrac{dx}{dt} = -\dfrac{x^2}{400} \cdot \sec^2(\theta) \cdot \dfrac{d\theta}{dt} = \dfrac{800}{2} = \boxed{400}$

3. (20 points) Consider the Mean Value Theorem:

   (a) State the conditions and the conclusion of the MVT.

   (b) Find all numbers c that satisfy the conclusion of the MVT for the function $f(x) = \dfrac{1}{x}$ over the interval $[1, 3]$.

   Solution:

(a) If $f(x)$ is a continuous functional relationship on $[a, b]$ and $f(x)$ is differentiable on (a, b), then $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ for $a < c < b$

(b) $\dfrac{\frac{1}{3} - \frac{1}{1}}{3 - 1} = -\dfrac{1}{c^2} \implies -\dfrac{1}{3} = -\dfrac{1}{c^2} \implies c^2 = 3 \implies c = \pm\sqrt{3}$, so $\boxed{c = \sqrt{3}}$

4. (20 points) Suppose x represents the edge length of a metal cube.

(a) If $V(x) = x^3$ is the volume of the cube, then find dV, the differential of V.

(b) Suppose the edge of the cube was originally found to be 10 cm in length but expands due to heat to 10.01 cm, use differentials to estimate the change in the volume, ΔV.

Solution:

(a) $V = x^3 \implies \dfrac{dV}{dx} = 3x^2 \implies \boxed{dV = 3x^2 dx}$

(b) $dV = 3(10)^2(0.01) = 3 \cdot 100 \cdot \frac{1}{100} = \boxed{3}$

5. (18 points) Find the absolute maximum and minimum values of $f(x) = 3x^{\frac{2}{3}} - 2x$ on the interval $[-1, 1]$. Be sure to list the requested optimum values; not their location nor the point (x, y).

Solution: $f'(x) = 2x^{-\frac{1}{3}} - 2$
$f'(x) = 0 \implies x = 1$
$f'(x) = \emptyset \implies x = 0$
$f(-1) = 5,\ f(0) = 0,\ f(1) = 1.$
therefore, $\boxed{\text{5 is the absolute maximum value of f(x)}}$, and $\boxed{\text{0 is the absolute minimum value of f(x)}}$.

6. (18 points) Find the following limits (be sure to show your work for full credit):

(a) $\displaystyle\lim_{n\to\infty} \left[\frac{-18}{n^2} \cdot \frac{n(n+1)}{2}\right]$

(b) $\displaystyle\lim_{x\to 4^+} \left[\frac{12 - 3x}{|x - 4|}\right]$

Solution:

(a) $\displaystyle\lim_{n\to\infty} \left[\frac{-18n^2 - 18n}{2n^2}\right] = \lim_{n\to\infty} \left[\frac{-18 - \frac{18}{n}}{2}\right] = -\frac{18}{2} = \boxed{-9}$

(b) $\displaystyle\lim_{x\to 4^+} \left[\frac{12 - 3x}{|x - 4|}\right] = \lim_{x\to 4^+} \left[\frac{3(4 - x)}{|x - 4|}\right] = \lim_{x\to 4^+} \begin{cases} \frac{3(4-x)}{x-4}, & x - 4 > 0 \\ \frac{3(4-x)}{-(x-4)}, & x - 4 < 0 \end{cases} =$

$\displaystyle\lim_{x\to 4^+} \begin{cases} -3, & x > 4 \\ 3, & x < 4 \end{cases} = \lim_{x\to 4^+} [-3] = \boxed{-3}$

7. (24 points) Find the following derivatives (be sure to simplify for full credit):

(a) Find $f'(x)$ given that $f(x) = (\sqrt{2x+3})^{100}$

(b) Find y'' where $y = \dfrac{2x^4 + 5x^3 - 18x^2}{6x - 12}$

(c) Find the derivative, y', of the following: $y = \dfrac{3x+2}{2x+3}$

Solution:

(a) $f'(x) = 100(\sqrt{2x+3})^{99} \cdot \frac{1}{2}(2x+3)^{-\frac{1}{2}}(2) = \frac{100(\sqrt{2x+3})^{99}}{\sqrt{2x+3}} = 100(\sqrt{2x+3})^{98} = \boxed{100(2x+3)^{49},\ x \geq -\frac{3}{2}}$

(b) $y = \dfrac{x^2(2x^2+5x-18)}{6(x-2)} = \dfrac{x^2(2x+9)(x-2)}{6(x-2)} = \dfrac{2x^3+9x^2}{6} = \dfrac{1}{3}x^3 + \dfrac{3}{2}x^2$

$y' = x^2 + 3x$

$\boxed{y'' = 2x + 3}$

(c) $y' = \dfrac{(2x+3)\cdot 3 - (3x+2)\cdot 2}{(2x+3)^2} = \boxed{\dfrac{5}{(2x+3)^2}}$

8. (16 points) Find the following derivatives, show your work, but only put a multiple choice letter as your final answer:

I. Find $\dfrac{dy}{dx}$ given that $y \cdot \cos(x) = x^2 + y^2$.

A. $y' = \dfrac{2x}{2y + \sin(x)}$

B. $y' = \dfrac{2x + y \cdot \sin(x)}{\cos(x) - 2y}$

C. $y' = \dfrac{2x + 2yy'}{-\sin(x)}$

D. $y' = \dfrac{2x - 2yy'}{-\sin(x)}$

E. $\dfrac{\cos(x)\cdot 2x + (x^2+y^2)\sin(x)}{\cos^2(x)(1 - 2y\cos(x))}$

F. None of the Above

II. Find $\dfrac{dy}{dx}$ if $y = x^2 \sec(x)\tan(x)$

A. $x^2\sec^3(x) + x^2\sec(x)\tan^2(x) + 2x\sec(x)\tan(x)$

B. $2x^2\sec(x)\tan^2(x) + 2x\sec(x)\tan(x)$

C. $x^2\sec(x) + 2x\sec(x)\tan(x)$

D. $2x\sec(x)\tan(x)\sec^2(x)$

E. $2x\sec(x)$

F. None of the Above

Solution:

I. B.

II. A.