INSTRUCTIONS: Outside paper and electronic devices are **not** permitted. Exam is worth 100 points. Neatness counts. Unless indicated, answers with no supporting work may receive no credit. $\boxed{\text{BOX}}$ your final answers.

1. (18 points) Consider the functions $g(x) = 2 - 2x$ and $h(x) = x^2 - 5x + 4$

   (a) Find the equation of the secant line through $h(0)$ and $h(2)$. Write your answer in slope-intercept form.

   (b) Find the equation of the tangent line through $h(0)$. Write your answer in slope-intercept form.

   (c) Find $(g/h)'(x)$.

   Solution:

   (a) $h(0) = 4$ and $h(2) = -2$. $m = \dfrac{4-(-2)}{0-2} = \dfrac{6}{-2} = -3$
   $y - 4 = -3(x-0)$, or $\boxed{y = -3x + 4}$

   (b) $h(0) = 4$, $h'(x) = 2x - 5 \implies h'(0) = -5$.
   $y - 4 = -5(x-0)$, or $\boxed{y = -5x + 4}$

   (c) $\left(\dfrac{g}{h}\right)(x) = \dfrac{2(1-x)}{(x-4)(x-1)} = \dfrac{-2}{x-4}$
   $\left(\dfrac{g}{h}\right)'(x) = \dfrac{(x-4)\cdot 0 + 2\cdot 1}{(x-4)^2} = \boxed{\dfrac{2}{(x-4)^2}}$

2. (18 points) Consider the function $f(x) = \sqrt{x+2}$.

   (a) What is the average rate of change of $f(x)$ on $[7, 14]$?

   (b) What is the instantaneous rate of change of $f(x)$ at $x = 7$?

   (c) Evaluate the expression: $\lim\limits_{x\to 7}\left[\dfrac{f(x) - f(7)}{x-7}\right]$.

   Solution:

   (a) $f(14) = 3$ and $f(7) = 3$. $m = \dfrac{4-3}{14-7} = \boxed{\dfrac{1}{7}}$

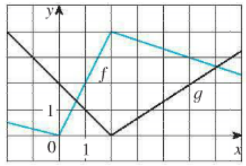
   (b) $f(7) = 3$, $f'(x) = \dfrac{1}{2}\cdot(x+2)^{-\frac{1}{2}} \implies f'(7) = \boxed{\frac{1}{6}}$

   (c) $\lim\limits_{x\to 7}\left[\dfrac{\sqrt{x+2}-3}{x-7}\right] = \lim\limits_{x\to 7}\left[\dfrac{\sqrt{x+2}-3}{x-7}\cdot\dfrac{\sqrt{x+2}+3}{\sqrt{x+2}+3}\right] = \lim\limits_{x\to 7}\left[\dfrac{x+2-9}{(x-7)(\sqrt{x+2}+3)}\right] =$
   $\lim\limits_{x\to 7}\left[\dfrac{1}{\sqrt{x+2}+3}\right] = \boxed{\dfrac{1}{6}}$

3. (15 points) Consider the function $f(x) = \frac{1}{\sqrt{x}}$. Find $f'(x)$ with the limit definition of derivative.

Solution: $f'(x) = \lim_{h\to 0}\left[\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}\right] = \lim_{h\to 0}\left[\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\right] = \lim_{h\to 0}\left[\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right] =$
$\lim_{h\to 0}\left[\frac{x-x-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\right] = \lim_{h\to 0}\left[\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\right] = \boxed{-\frac{1}{2x\sqrt{x}}}$

4. (18 points) The following questions are not necessarily related.

(a) Consider the following graph: 

Let $u(x) = (f\circ g)(x)$, what is $u'(1)$?

(b) $y = \left[\frac{x\cot(2x)}{5}\right] \implies \frac{dy}{dx} = ?$

(c) Given $y = \left[\frac{|2x-2|}{x-1}\right]$, graph y and find $\frac{dy}{dx}$.

Solution:

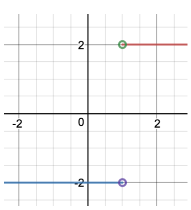
(a) $(f\circ g)'(x) = f'(g(x))\cdot g'(x)$, therefore, $u'(x) = f'(g(1))\cdot g'(1) = f'(1)\cdot g'(1) = 2\cdot(-1) = \boxed{-2}$

(b) $y = \frac{1}{5}x\cdot\cot(2x) \implies y' = \frac{1}{5}x\cdot(-1)\csc^2(2x)\cdot 2 + \cot(2x)\cdot\frac{1}{5} = \boxed{\frac{-2x\csc^2(2x)+\cot(2x)}{5}}$

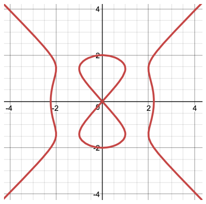
(c) $y = \begin{cases} \frac{2(x-1)}{(x-1)} & , x-1>0 \\ -\frac{2(x-1)}{(x-1)} & , x-1<0 \end{cases}$

$y = \begin{cases} 2 & , x>1 \\ -2 & , x<1 \end{cases}$

$\boxed{y' = \begin{cases} 0 & , x>1 \\ 0 & , x<1 \end{cases}}$



5. (15 points) Consider the graph of $y^2(y^2-4) = x^2(x^2-5)$ seen below. Find $\dfrac{dy}{dx}$ and evaluate it at $(1, \sqrt{2})$.



Solution: $y^2(y^2-4) = x^2(x^2-5) \implies y^4 - 4y^2 = x^4 - 5x^2 \implies 4y^3 \cdot y' - 8yy' = 4x^3 - 10x \implies y'(4y^3 - 8y) = 4x^3 - 10x \implies y' = \dfrac{2x(2x^2-5)}{4y(y^2-2)}$

$$y'(1,\sqrt{2}) = \frac{2(1)(2\cdot 1^2 - 5)}{4\sqrt{2}(2-2)} = \boxed{\emptyset}$$

6. (16 points) Only the answer will be graded on the following questions; no work is required.

I. Where is $y = \dfrac{3x^2 - x - 4}{x^2 - 1}$ non-differentiable?

A. $x = 1$, $x = -1$

B. $x = 1$

C. $x = -1$

D. $x = -1$, $x = \dfrac{4}{3}$

E. $x = \dfrac{4}{3}$

F. Nowhere

II. Suppose the position of a moving object (in meters) at any time (in seconds) is given by the expression $s(t) = -4.5t^2 + 10t + 20$. What is the acceleration of the object after 10 seconds?

A. 4.5 m/s^2 upwards

B. 4.5 m/s^2 downwards

C. 9 m/s^2 upwards

D. 9 m/s^2 downwards

E. 0 m/s^2.

F. None of the above

III. Given that $y = x^3 \cdot \cos\left(\dfrac{2}{x}\right)$, find y'.

A. $2x \sin\left(\dfrac{2}{x}\right)$

B. $-3x^2 \sin\left(\dfrac{2}{x}\right)$

C. $6 \sin\left(\frac{2}{x}\right)$

D. $2x \sin\left(\dfrac{2}{x}\right) + 3x^2 \cos\left(\dfrac{2}{x}\right)$

E. $3x^2 \cos\left(\dfrac{2}{x}\right) - x^3 \sin\left(\dfrac{2}{x}\right)$

F. None of the above

IV. Find $f'(1)$, given $f(x) = \begin{cases} x & , x < 1 \\ 3 & , x = 1 \\ 2 - x^2 & , 1 < x \leq 2 \\ x - 3 & , x > 2 \end{cases}$.

A. 1

B. 3

C. 0

D. -2

E. D.N.E.

F. None of the above

Solution:

I. A

II. D

III. D

IV. E