INSTRUCTIONS: Outside paper and electronic devices are **not** permitted. Exam is worth 100 points. Neatness counts. Unless indicated, answers with no supporting work may receive no credit. $\boxed{\text{BOX}}$ your final answers.

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1. (24 points) Consider the functions $g(x) = 2 - 2x$ and $h(x) = x^2 - 5x + 4$

   (a) Consider $f(x) = 2 + \dfrac{g(x)}{h(x)}$. Find any discontinuities and label them as asymptotes or holes.
   List the (x, y) coordinates of any hole.
   Use the definition of vertical asymptote to show any asymptotic behavior.

   (b) Consider $k(x) = h(x) + 5x - 4 - \sqrt{-\frac{1}{2} \cdot g(x)}$.
   Use the IVT to show that there is a real number such that $k(x) = 4$.

   (c) Find $\lim\limits_{x\to\infty} [(g/h)(x)]$

   Solution: ANS:

   (a) $f(x) = 2 + \dfrac{2(1-x)}{(x-4)(x-1)} = \dfrac{2(x-5)}{(x-4)}$

   At $x = 1$ there is a hole. $f(1) = \dfrac{8}{3} \implies$ the coordinates of the point are $\boxed{\left(1, \dfrac{8}{3}\right)}$

   $\lim\limits_{x\to 4^+} \left[\dfrac{2(x-5)}{(x-4)}\right] = \dfrac{-2}{0^+} = -\infty \implies \boxed{y = 4}$ is a vertical asymptote.

   (b) $k(x) = x^2 - 5x + 4 + 5x - 4 - \sqrt{-1 + x} = x^2 - \sqrt{x-1}$
   $k(x)$ is continuous on $(1, \infty)$ and therefore on $(1, 5)$
   $k(1) = 1$, $k(5) = 23$ and $1 < 4 < 23$. So by the I.V.T. there exists a $1 < C < 5$ such that $f(C) = 4$.

   (c) $\lim\limits_{x\to\infty} \left[\dfrac{2-2x}{x^2-5x+4}\right] = \lim\limits_{x\to\infty} \left[\dfrac{2(1-x)}{(x-4)(x-1)}\right] = \lim\limits_{x\to\infty} \left[\dfrac{-2(-1+x)}{(x-4)(x-1)}\right] = \lim\limits_{x\to\infty} \left[\dfrac{-2(x-1)}{(x-4)(x-1)}\right] =$

   $\lim\limits_{x\to\infty} \left[\dfrac{-2}{x-4}\right] = \lim\limits_{x\to\infty} \left[\dfrac{-2}{x-4} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}}\right] = \lim\limits_{x\to\infty} \left[\dfrac{\frac{-2}{x}}{1 - \frac{4}{x}}\right] = \dfrac{0}{1-0} = \boxed{0}$

2. (10 points) Show that the function $f(x) = \begin{cases} |x^3 + 1| & , x \geq -1 \\ -x - 1 & , x < -1 \end{cases}$ is continuous by definition.

   Solution: ANS: $f(x)$ is made of pieces that are continuous on the real numbers since the pieces are polynomials or absolute value of polynomials.
   We need only check for a discontinuity at $x = -1$.

   $f(x)$ is continuous at $x = -1$ so long as $f(-1) = \lim\limits_{x\to -1} [f(x)]$

   $f(-1) = 0$ and $\lim\limits_{x\to -1^-} [(x)] = 0$ and $\lim\limits_{x\to -1^+} [(x)] = 0$. Therefore, $\lim\limits_{x\to -1} [(x)] = 0$

   i.e. $f(-1) = \lim\limits_{x\to -1} [f(x)]$

3. (24 points) Evaluate the following limits:

(a) $\lim_{x\to 1}\left[\frac{|2x-2|}{x-1}\right] = ?$

(b) $\lim_{h\to 0}\left[\frac{\sqrt{9+h}-3}{h}\right] = ?$

(c) $\lim_{x\to 0^-}\left[x^3\cos\left(\frac{2}{x}\right)\right] = ?$

(d) $\lim_{x\to 0}\left[\frac{\sin(3x)\sin(5x)}{x^2}\right] = ?$

Solution:

(a) Since $\frac{2|x-1|}{x-1} = \begin{cases} 2 & x>1 \\ -2 & x<1 \end{cases}$

We have $\lim_{x\to 1^-}[-2] = -2$ and $\lim_{x\to 1^+}[2] = 2$. Therefore, $\lim_{x\to 1}\left[\frac{2|x-1|}{x-1}\right] = \boxed{\text{D.N.E.}}$

(b) $\lim_{h\to 0}\left[\frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right] = \lim_{h\to 0}\left[\frac{9+h-9}{h(\sqrt{9+h}+3)}\right] = \lim_{h\to 0}\left[\frac{1}{\sqrt{9+h}+3}\right] = \boxed{\frac{1}{6}}.$

(c) We know that $-1 \le \cos\left(\frac{2}{x}\right) \le 1 \implies -x^3 \le x^3\cos\left(\frac{2}{x}\right) \le x^3$

also $\lim_{x\to 0^-}[-x^3] = 0$ and $\lim_{x\to 0^-}[x^3] = 0$.

Therefore, by the Squeeze Theorem we know $\lim_{x\to 0^-}\left[x^3\cos\left(\frac{2}{x}\right)\right] = 0.$

(d) $\lim_{x\to 0}\left[\frac{\sin(3x)\sin(5x)}{x^2}\cdot\frac{3}{3}\cdot\frac{5}{5}\right] = \lim_{x\to 0}\left[\frac{\sin(3x)}{3x}\cdot\frac{\sin(5x)}{5x}\cdot 3\cdot 5\right] = 1\cdot 1\cdot 3\cdot 5 = \boxed{15}.$

4. (18 points) Consider the function: $f(x) = \begin{cases} x & x<1 \\ 3 & x=1 \\ 2-x^2 & 1<x\le 2 \\ x-3 & x>2 \end{cases}$

(a) $\lim_{x\to 1^-} f(x) = ?$

(b) $\lim_{x\to 1} f(x) = ?$

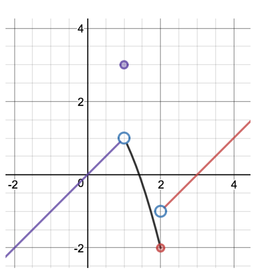
(c) $f(1) = ?$

(d) $\lim_{x\to 2^-} f(x) = ?$

(e) $\lim_{x\to 2^+} f(x) = ?$

(f) $\lim_{x\to 2} f(x) = ?$

Solution:



(a) $\lim_{x\to 1^-}[x] = \boxed{1}$

(b) $\lim_{x\to 1^-}[x] = \lim_{x\to 1^+}[2-x^2] = \boxed{1}$

(c) $f(1) = \boxed{3}$

(d) $\lim_{x\to 2^-}[2-x^2] = \boxed{-2}$

(e) $\lim_{x\to 2^+}[x-3] = \boxed{-1}$

(f) $\lim_{x\to 2^-}[2-x^2] \neq \lim_{x\to 2^+}[x-3] \implies \lim_{x\to 2}[f(x)] = \boxed{\text{D.N.E.}}$

5. (12 points) Consider the function $f(x) = \sqrt{x+2}$.

(a) Find the following limit: $\lim_{h\to 0} \left[\frac{f(7+h)-f(7)}{h}\right]$.

(b) Find the average rate of change of $f(x)$ on the interval $[7, 14]$

Solution:

(a) $\lim_{h\to 0} \left[\frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right] = \lim_{h\to 0} \left[\frac{9+h-9}{h(\sqrt{9+h}+3)}\right] = \lim_{h\to 0} \left[\frac{1}{\sqrt{9+h}+3}\right] = \boxed{\frac{1}{6}}$.

(b) Average R.O.C.$= \frac{f(14)-f(7)}{14-7} = \frac{4-3}{7} = \boxed{\frac{1}{7}}$

TURN OVER EXAM, MORE ON THE BACK

6. (12 points) No work is required for the following questions. Choose the most appropriate answer for each:

I. Suppose CompanyA asks ManufacturerB to provide barstock of length 4.5 inches with an acceptable tolerance of lengths between 4.35 and 4.65 inches. If length, L, is dependent upon input value n (for $n > 0$), and $L(n) = \sqrt{2n^2}$, then which of the following is the range of acceptable input values for n?

A. $0.30 \div 2$

B. $4.5 \pm \epsilon$

C. $\dfrac{4.5}{|n|\sqrt{2}} \pm \delta$

D. $\left(\dfrac{4.35}{\sqrt{2}}, \dfrac{4.65}{\sqrt{2}}\right)$

E. None of the above

II. $\lim_{x \to -1} \left[\dfrac{3x^2 - x - 4}{x^2 - 1}\right] = ?$

A. 3.5

B. D.N.E.

C. ∞

D. $-\frac{7}{2}$

E. None of the above

III. Describe the end behavior of $f(x) = \dfrac{\sqrt{2x^2+1}}{3x - 5}$. Choose ALL that apply.

A. $x \to \dfrac{5}{3}$

B. $y \to \dfrac{5}{3}$

C. $x \to \dfrac{\sqrt{2}}{3}$

D. $y \to \dfrac{\sqrt{2}}{3}$

E. $x \to -\dfrac{5}{3}$

F. $y \to -\dfrac{5}{3}$

G. $x \to -\dfrac{\sqrt{2}}{3}$

H. $y \to -\dfrac{\sqrt{2}}{3}$

IV. $\lim_{x \to \infty} \left[\dfrac{\sin^2(x)}{x^2}\right] = ?$

A. 0

B. 1

C. ∞

D. $-\infty$

E. D.N.E.

F. None of the above

Solution:

I. D

II. A

III. D & H

IV. A