INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **justify all answers**. A correct answer with incorrect work or no justification may receive no credit. Books, notes, electronic devices, other unauthorized devices, and help from another person are not permitted while taking the exam. The final exam is worth 150 points.

NOTE: YOU MAY TEAR OFF THIS FIRST PAGE AND USE (FRONT AND BACK) AS SCRATCH PAPER.

i. DO NOT START UNTIL INSTRUCTED BY A PROCTOR.

ii. THE EXAM IS ON BOTH SIDES OF EACH FOLLOWING EXAM PAGE

iii. WRITE YOUR NAME ON THE NEXT PAGE.

iv. WHEN YOU FINISH (IF BEFORE THE EXAM END TIME) PLEASE QUIETLY COLLECT YOUR THINGS AND FOLLOW PROCTOR INSTRUCTIONS IN UPLOADING YOUR EXAM WITH SUPPORTING WORK TO GRADESCOPE. ONLY WORK THAT'S SUBMITTED TO GRADESCOPE WILL BE GRADED.

Formulas that may be useful:

1. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
2. Circle: $(x-h)^2 + (y-k)^2 = r^2$
3. Arc length: $s = r\theta$
4. $\sin(a-b) = \sin a \cos b - \sin b \cos a$
5. $\cos(a-b) = \cos a \cos b + \sin a \sin b$
6. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
7. $\cos(2\theta) = 2\cos^2\theta - 1$
8. $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$
9. $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
10. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
11. Area of a sector: $A = \frac{1}{2}r^2\theta$
12. $\sin(a+b) = \sin a \cos b + \sin b \cos a$
13. $\cos(a+b) = \cos a \cos b - \sin a \sin b$
14. $\sin(2\theta) = 2\sin\theta\cos\theta$
15. $\cos(2\theta) = 1 - 2\sin^2\theta$
16. $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$
17. $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

Name: ____

1. Re-arrange the following list of numbers to go from smallest first to largest last: (4 pts)

$e^2,\ \sqrt{2},\ \frac{1}{\frac{1}{120}},\ \sqrt{3},\ |3-\pi|,\ 2^2,\ |-10|$

Solution:

$$\boxed{|3-\pi| < \sqrt{2} < \sqrt{3} < 2^2 < e^2 < |-10| < \frac{1}{\frac{1}{120}}}$$

2. Simplify the following: (12 pts)

(a) $\frac{1}{3} - \frac{\frac{5}{2}}{10} + 2^{-1}$

Solution:

$$\begin{aligned}
\frac{1}{3} - \frac{\frac{5}{2}}{10} + 2^{-1} &= \frac{1}{3} - \frac{5}{20} + \frac{1}{2} && (1)\\
&= \frac{20}{60} - \frac{15}{60} + \frac{30}{60} && (2)\\
&= \frac{35}{60} && (3)\\
&= \boxed{\frac{7}{12}} && (4)
\end{aligned}$$

(b) $\left(9x^3y^{-5}\right)\left(\frac{1}{3}x^{-5}\left(zy^2\right)^3 z^0\right)$

Solution:

$$\begin{aligned}
\left(9x^3y^{-5}\right)\left(\frac{1}{3}x^{-5}\left(zy^2\right)^3 z^0\right) &= \left(9x^3y^{-5}\right)\left(\frac{1}{3}x^{-5}z^3y^6\right) && (5)\\
&= 3x^{-2}y^1z^2 && (6)\\
&= \boxed{\frac{3yz^3}{x^2}} && (7)
\end{aligned}$$

(c) $\left(\frac{x^{-1/3}y^{1/2}}{x^{1/3}y^{1/4}}\right)^{-1}$

Solution:

$$\begin{aligned}
\left(\frac{x^{-1/3}y^{1/2}}{x^{1/3}y^{1/4}}\right)^{-1} &= \left(x^{-\frac{2}{3}}y^{\frac{1}{4}}\right)^{-1} && (8)\\
&= x^{\frac{2}{3}}y^{-\frac{1}{4}} && (9)\\
&= \boxed{\frac{x^{\frac{2}{3}}}{y^{\frac{1}{4}}}} && (10)
\end{aligned}$$

(d) $\sqrt{12a^2y}$

Solution:

$$\sqrt{12a^2y} = \sqrt{2^2\cdot 3\cdot a^2y} = \boxed{2|a|\sqrt{3y}}$$

3. Simplify the following: (8 pts)

   (a) $\log_4(16) - 10^{2\log(6)} + \ln(1)$

   Solution:

$$\begin{aligned} \log_4(16) - 10^{2\log(6)} + \ln(1) &= \log_4(4^2) - 10^{\log(6^2)} + 0 && (11)\\ &= 2 - 6^2 && (12)\\ &= \boxed{-34} && (13) \end{aligned}$$

   (b) $\dfrac{1 - \frac{1}{x-1}}{\frac{1}{x-1} + 1}$

   Solution:

$$\begin{aligned} \frac{1 - \frac{1}{x-1}}{\frac{1}{x-1} + 1} &= \frac{\frac{(x-1)}{x-1} - \frac{1}{x-1}}{\frac{1}{x-1} + \frac{x-1}{x-1}} && (14)\\ &= \frac{\left(\frac{x-2}{x-1}\right)}{\left(\frac{1+x-1}{x-1}\right)} && (15)\\ &= \left(\frac{x-2}{x-1}\right)\left(\frac{x-1}{x}\right) && (16)\\ &= \boxed{\frac{x-2}{x}} && (17) \end{aligned}$$

4. Solve the following equations for x: (10 pts)

   (a) $(x-4)^2 = 5$
   Solution:

$$\begin{aligned} (x-4)^2 &= 5 && (18)\\ (x-4) &= \pm\sqrt{5} && (19)\\ x &= \boxed{4 \pm \sqrt{5}} && (20) \end{aligned}$$

   (b) $\dfrac{3}{x-2} + \dfrac{2}{x+2} = 1$

   Solution:

$$\begin{aligned} (x-2)(x+2)\left(\frac{3}{x-2} + \frac{2}{x+2}\right) &= 1(x-2)(x+2) && (21)\\ 3(x+2) + 2(x-2) &= x^2 - 4 && (22)\\ 5x + 2 &= x^2 - 4 && (23)\\ 0 &= x^2 - 5x - 6 && (24)\\ 0 &= (x-6)(x+1) && (25) \end{aligned}$$

   This results in potential answers $x = -1, 6$. Checking these values in the original equation we see that both values do solve the original equation.Hence the solution is $\boxed{x = -1, 6}$

5. Solve the following equations for x: (10 pts)

(a) $\log_2(x) = 1 - \log_2(x-1)$

Solution:

$$\log_2(x) = 1 - \log_2(x-1) \tag{26}$$
$$\log_2(x) + \log_2(x-1) = 1 \tag{27}$$
$$\log_2(x(x-1)) = 1 \tag{28}$$
$$x(x-1) = 2 \tag{29}$$
$$x^2 - x - 2 = 0 \tag{30}$$
$$(x-2)(x+1) = 0 \tag{31}$$

This results in potential answers $x = -1, 2$. However, $x = -1$ is not in the domain of $\log_2(x)$. On the other hand, we check that $x = 2$ indeed solves the original equation. Hence the solution is $\boxed{x = 2}$

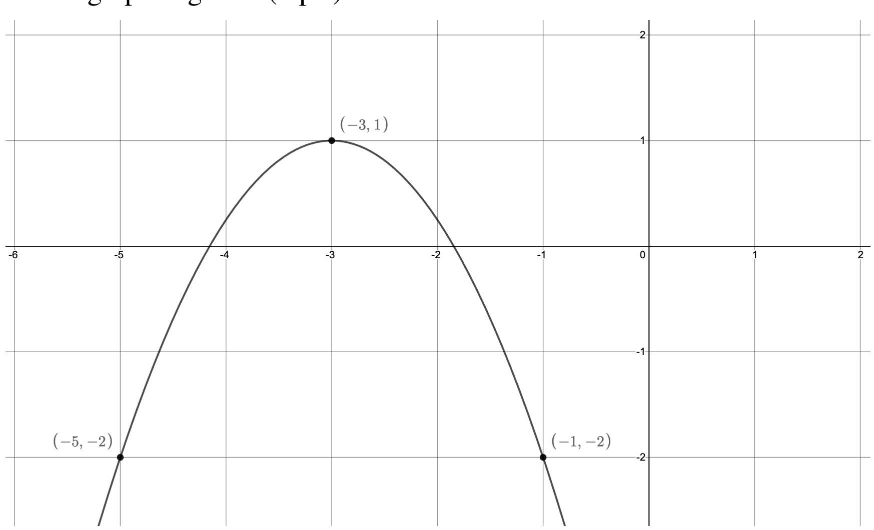
(b) $1 = x + \sqrt{3-x}$

Solution:

$$1 = x + \sqrt{3-x} \tag{32}$$
$$1 - x = \sqrt{3-x} \tag{33}$$
$$(1-x)^2 = 3 - x \tag{34}$$
$$1 - 2x + x^2 = 3 - x \tag{35}$$
$$x^2 - x - 2 = 0 \tag{36}$$
$$(x-2)(x+1) = 0 \tag{37}$$

This results in potential answers $x = -1, 2$. Checking these values in the original equation, we see that only $x = -1$ solves the original equation. Hence the solution is $\boxed{x = -1}$

6. The following graph has equation of the form $y = a(x-h)^2 + k$. Find the equation of the function whose graph is given. (5 pts)



Solution: From the graph the vertex is located at $(-3, 1)$. The vertex of a parabola is found at (h, k) so we get $y = a(x - (-3))^2 + 1 = a(x + 3)^2 + 1$. To find a we can use one of the other given points. Plugging in $(-1, -2)$ we get $-2 = a(-1 + 3)^2 + 1$ and solving for a we find:

$$-2 = a(-1+3)^2 + 1 \tag{38}$$
$$-2 = 4a + 1 \tag{39}$$
$$-3 = 4a \tag{40}$$
$$-\frac{3}{4} = a \tag{41}$$

Resulting in equation: $\boxed{y = -\frac{3}{4}(x+3)^2 + 1}$

7. Consider the function $f(x) = \dfrac{x}{4x^3 - 16x}$. Answer the following: (12 pts)

   (a) Find the domain of $f(x)$

   Solution: The domain is all real numbers except when $4x^3 - 16x = 0$. Solving this we get:

$$4x^3 - 16x = 0 \tag{42}$$
$$4x(x^2 - 4) = 0 \tag{43}$$
$$4x(x-2)(x+2) = 0 \tag{44}$$

   Resulting in values $x = -2, 0, 2$ that must be removed from the domain. The domain is: $\boxed{(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)}$.

   (b) Determine whether $f(x)$ is odd, even, or neither. Justify your answer.

   Solution: Beginning with $f(-x)$ we get:

$$f(-x) = \frac{(-x)}{4(-x)^3 - 16(-x)} = \frac{-x}{-4x^3 + 16x} = \frac{-x}{-\left(4x^3 - 16x\right)} = \frac{x}{4x^3 - 16x} = f(x)$$

   So $f(x)$ is $\boxed{\text{even}}$.

   (c) Find x-coordinates of any hole(s). If there are none write NONE.

   Solution: $\dfrac{x}{4x^3 - 16x} = \dfrac{x}{4x\left(x^2 - 4\right)} = \dfrac{1}{4\left(x^2 - 4\right)}$ so there is a hole at $\boxed{x = 0}$

   (d) Find y-coordinates of any hole(s). If there are none write NONE.

   Solution:: The y-coordinate is found by plugging $x = 0$ into $\dfrac{1}{4\left(x^2 - 4\right)}$ and we get $\dfrac{1}{4\left(0^2 - 4\right)} = -\dfrac{1}{16}$. So the hole is located at $\boxed{y = -\frac{1}{16}}$

   (e) Find any horizontal/slant asymptotes. If there are none write NONE.

   Solution: There is a horizontal asymptote of $\boxed{y = 0}$ since $\dfrac{x}{4x^3 - 16x} \to \dfrac{x}{4x^3} = \dfrac{1}{4x^2} \to 0$ as $x \to \pm\infty$.

8. A hot cup of coffee is placed in a 70° F dining room. The temperature of the coffee, T in degrees Fahrenheit, as a function of time, t in minutes, is modeled by Newton's law of cooling and is given by the function: $T(t) = 70 + 130e^{-0.05t}$. (11 pts)

(a) Find the temperature of the coffee after 100 minutes. Leave your answer in exact form, **do not** attempt to approximate with a decimal value

Solution:

$$T(100) = 70 + 130e^{-0.05(100)} \quad (45)$$
$$= 70 + 130e^{-5} \quad (46)$$

Hence the temperature of the coffee after 100 minutes is $\boxed{(70 + 130e^{-5})^\circ F}$

(b) How long until the coffee reaches 110 ° F? Leave your answer in exact form, **do not** attempt to approximate with a decimal value.

Solution:

Since the final temperature is $T(t) = 110$, we have

$$110 = 70 + 130e^{-0.05t} \quad (47)$$
$$40 = 130e^{-0.05t} \quad (48)$$
$$\frac{40}{130} = e^{-0.05t} \quad (49)$$
$$\ln\left(\frac{4}{13}\right) = -0.05t \quad (50)$$
$$-\frac{1}{0.05}\ln\left(\frac{4}{13}\right) = t \quad (51)$$

which gives the time as $\boxed{20\ln\left(\frac{4}{13}\right) \text{ minutes}}$

(c) According to the model, what is the initial temperature of the coffee?

Solution:

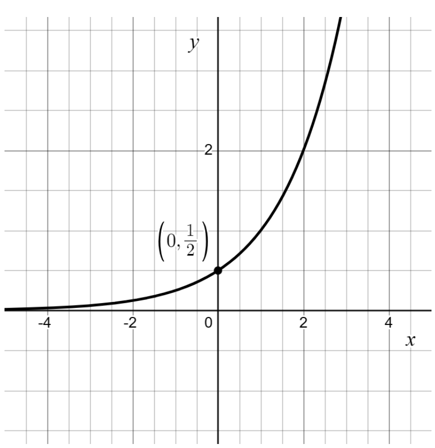
The initial temperature occurs when $t = 0$:

$$T(0) = 70 + 130e^{-0.05(0)} \quad (52)$$
$$= 70 + 130 \quad (53)$$
$$= 200 \quad (54)$$

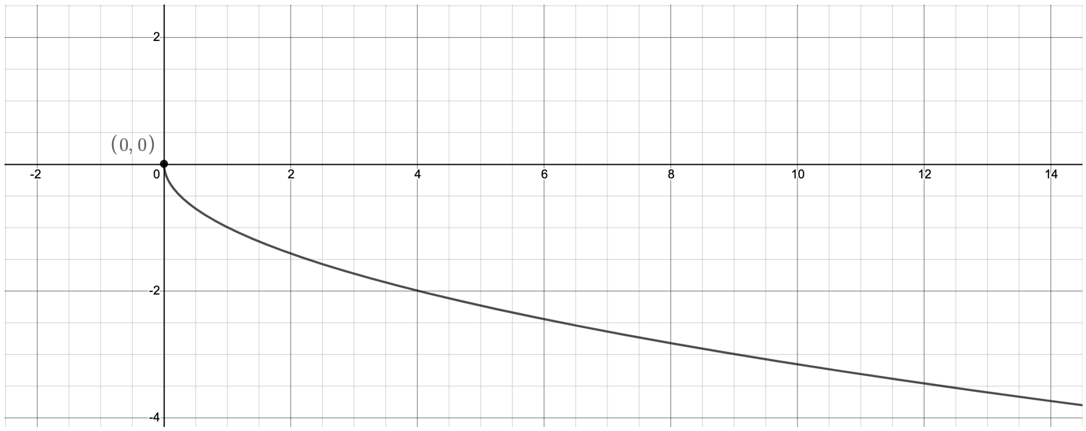
Hence the initial temperature of the coffee is $\boxed{200^\circ F}$

9. Sketch the graph of each of the following. Make sure to label relevant value(s) on your axe(s). (12 pts)

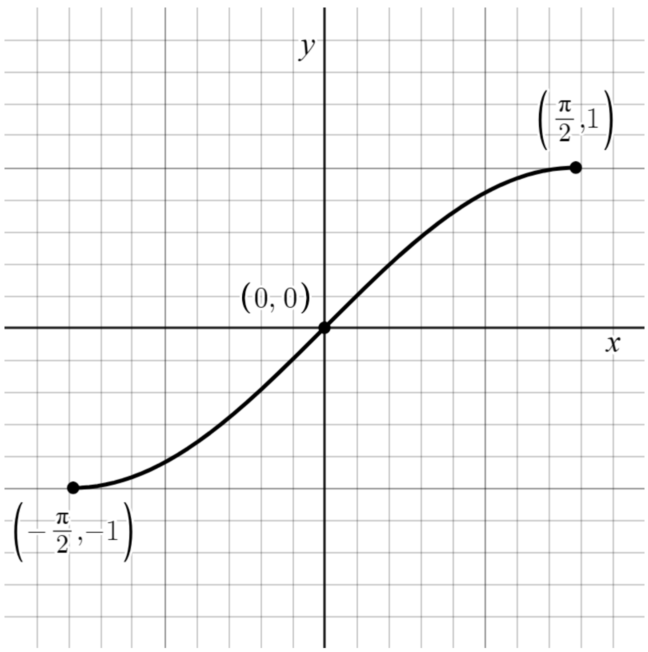
(a) $f(x) = 2^{x-1}$



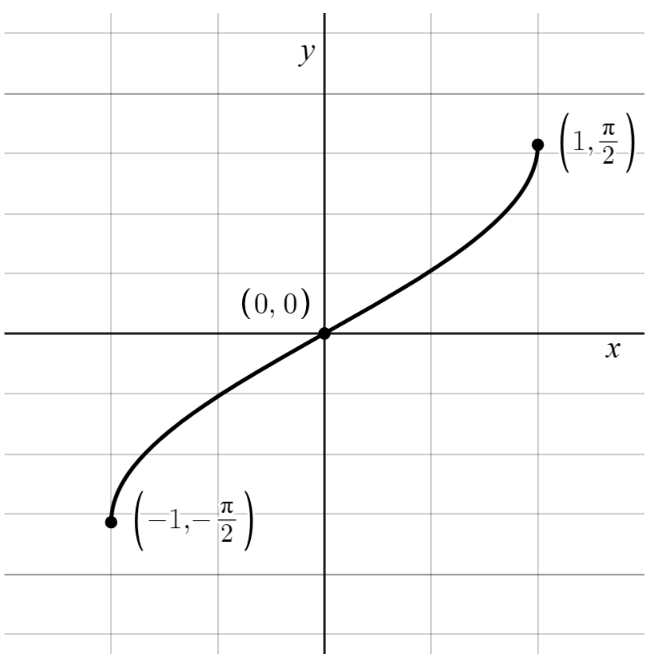
(b) $g(x) = -\sqrt{x}$



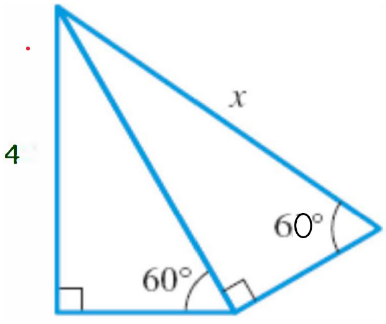
(c) $h(x) = \sin x$ on the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(d) $i(x) = \sin^{-1} x$

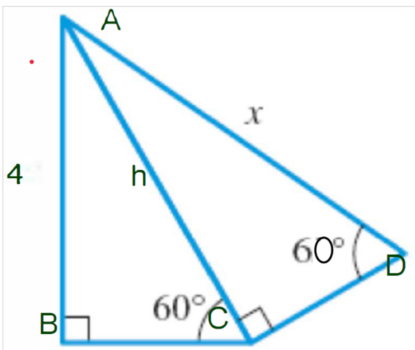


10. Solve for x as depicted in the image (6 pts)



Solution:

Let us first name the vertices and one of the sides of the triangles



From triangle ABC, we have

$$\sin 60^o = \frac{4}{h} \tag{55}$$

$$h = \frac{4}{\sin 60^o} \tag{56}$$

Now, from triangle ACD, we have

$$\sin 60^o = \frac{h}{x} \tag{57}$$

$$x = \frac{h}{\sin 60^o} \tag{58}$$

$$= \frac{\left(\frac{4}{\sin 60^o}\right)}{\sin 60^o} \tag{59}$$

$$= \frac{4}{\sin^2 60^o} \tag{60}$$

$$= \frac{4}{\left(\frac{3}{4}\right)} \tag{61}$$

Hence the solution is $\boxed{x = \frac{16}{3}}$

11. Find the exact value: (15 pts)

(a) $\cos\left(\frac{7\pi}{6}\right)$

Solution:

$$\boxed{\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}}$$

(b) $\csc\left(\frac{4\pi}{3}\right)$

Solution:

$$\boxed{\csc\left(\frac{4\pi}{3}\right) = -\frac{2}{\sqrt{3}}}$$

(c) $\tan^{-1}(-1)$

Solution:

$$\boxed{\tan^{-1}(-1) = -\frac{\pi}{4}}$$

(d) $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

Solution:

$$\boxed{\arccos\left(\cos\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}}$$

(e) $\sec\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

Solution:

$$\boxed{\sec\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{2}{\sqrt{3}}}$$

12. Verify the identity: $\cos^2\theta\cot\theta = \cot\theta - \cos\theta\sin\theta$ (5 pts)
(Hint: Start with the right hand side and recall the definition of $\cot\theta$)

Solution:

Starting with the Right Hand side

$$\cot\theta - \cos\theta\sin\theta = \frac{\cos\theta}{\sin\theta} - \cos\theta\sin\theta \tag{62}$$

$$= \cos\theta\left(\frac{1}{\sin\theta} - \sin\theta\right) \tag{63}$$

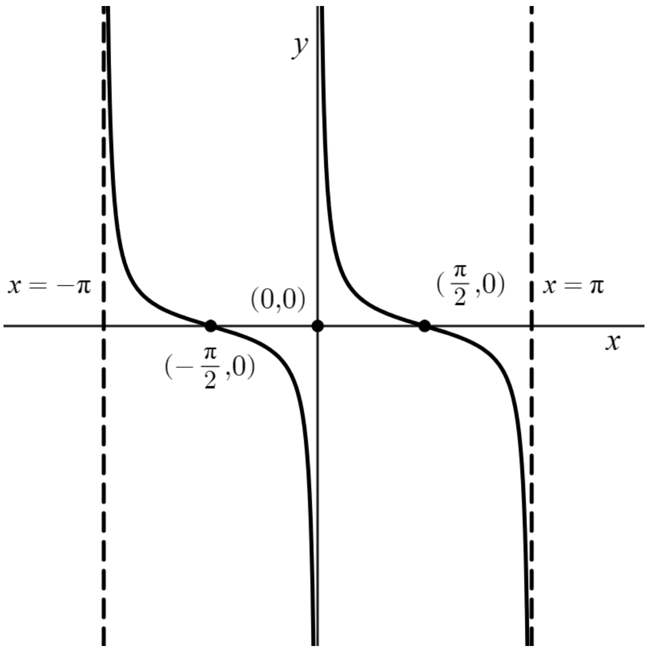
$$= \cos\theta\left(\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}\right) \tag{64}$$

$$= \cos\theta\left(\frac{1-\sin^2\theta}{\sin\theta}\right) \tag{65}$$

$$= \frac{\cos\theta\cos^2\theta}{\sin\theta} \tag{66}$$

$$= \boxed{\cos^2\theta\cot\theta} \tag{67}$$

13. Use the graph of the function $f(x)$ below, with $f(0) = 0$ and with domain $(-\pi, \pi)$, to answer the following questions. No justification of your answers is needed for this problem. (10 pts)



(a) Solve the equation $f(x) = 0$
Solution:

$\boxed{x = -\frac{\pi}{2}, 0, \frac{\pi}{2}}$

(b) Solve the inequality $f(x) < 0$ and give your answer in interval notation
Solution:

$\boxed{\left(-\frac{\pi}{2}, 0\right) \cup \left(\frac{\pi}{2}, \pi\right)}$

(c) Identify the restriction of domain of $f(x)$ so that the range is preserved and the graph is one-to-one
Solution:

$\boxed{(0, \pi)}$ Other acceptable answers are $\boxed{(-\pi, 0)}$ and $\boxed{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

(d) is $f(x)$ odd, even, or neither?
Solution:

$\boxed{\text{Odd}}$

14. Find all solutions to the following equations: (10 pts)

(a) $\cos(4\theta) = \frac{1}{2}$

Solution:
Using the Unit circle, we notice the following solutions (where k is any integer)

i.

$$4\theta = \frac{\pi}{3} + 2k\pi \tag{68}$$

$$\theta = \boxed{\frac{\pi}{12} + k\frac{\pi}{2}} \tag{69}$$

ii.

$$4\theta = \frac{5\pi}{3} + 2k\pi \tag{70}$$

$$\theta = \boxed{\frac{5\pi}{12} + k\frac{\pi}{2}} \tag{71}$$

(b) $\sin\theta\tan\theta - 2\tan\theta = 0$

Solution:

$$\sin\theta\tan\theta - \tan\theta = 0 \tag{72}$$

$$\tan\theta\left(\sin\theta - 1\right) = 0 \tag{73}$$

By multiplicative property of zero we get two equations(again, we will use k as any integer)

i. $\tan\theta = 0$

This has solutions $\theta = k2\pi$ and $\theta = \pi + k2\pi$. These can be combined into $\theta = k\pi$

ii.

$$\sin\theta - 2 = 0 \tag{74}$$

$$\sin\theta = 2 \tag{75}$$

This cannot happen because the range of $\sin\theta$ is $[-1, 1]$ Hence the only solutions are the ones found above, i.e $\boxed{\theta = k\pi}$

15. Find the exact value for each: (8 pts)

(a) $\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{12}\right) - \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{12}\right)$

Solution:

$$\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{12}\right) - \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{6} + \frac{\pi}{12}\right) \tag{76}$$

$$= \cos\left(\frac{\pi}{4}\right) \tag{77}$$

$$= \boxed{\frac{\sqrt{2}}{2}} \tag{78}$$

(b) $\sin\left(15^o\right)$

Solution:

First note that $\sin\left(15^o\right)$ is positive because $\left(15^o\right)$ is in Quadrant I.Hence,using the Half Angle formula, we can write//

$$\sin\left(15^o\right) = \sin\left(\frac{30^o}{2}\right) \tag{79}$$

$$= \sqrt{\frac{1 - \cos\left(30^o\right)}{2}} \tag{80}$$

$$= \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} \tag{81}$$

$$= \sqrt{\frac{2 - \sqrt{3}}{4}} \tag{82}$$

$$= \boxed{\frac{\sqrt{2 - \sqrt{3}}}{2}} \tag{83}$$

16. For $f(x) = 3\cos\left(x - \frac{\pi}{3}\right)$ (12 pts)

(a) Identify the amplitude.

Solution:
Comparing $f(x) = 3\cos\left(x - \frac{\pi}{3}\right)$ to $y = a\cos(bx + c)$, amplitude $= |a| = |3| = \boxed{3}$

(b) Identify the period.

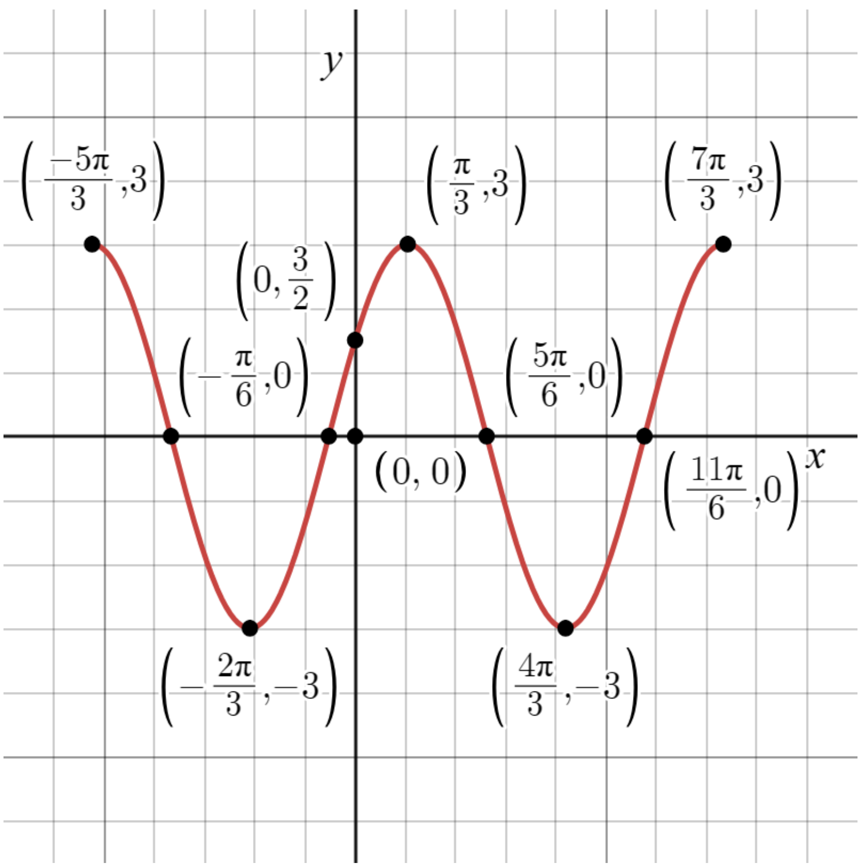
Solution:
Comparing $f(x) = 3\cos\left(x - \frac{\pi}{3}\right)$ to $y = a\cos(bx + c)$, period $= \frac{2\pi}{|b|} = \frac{2\pi}{|1|} = \boxed{2\pi}$

(c) Identify the phase shift.

Solution:
Comparing $f(x) = 3\cos\left(x - \frac{\pi}{3}\right)$ to $y = a\cos(bx + c)$, phase shift $= -\frac{c}{b} = -\frac{\left(-\frac{\pi}{3}\right)}{1} = \boxed{\frac{\pi}{3}}$

(d) Sketch two cycles of $f(x)$ labeling relevant y-values and at least two values on the x-axis.

Solution:
The step size is calculated by using $\frac{\text{period of } f(x)}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$. Utilizing the step size, amplitude, and phase shift we get the graph:



End of Final Exam. Have an awesome summer!