INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **justify all answers**. A correct answer with incorrect work or no justification may receive no credit. Books, notes, electronic devices, other unauthorized devices, and help from another person are not permitted while taking the exam. The exam is worth 100 points.

Potentially useful formulas:

1. $\log_b(u) = \dfrac{\log_a(u)}{\log_a(b)}$
2. $A = \dfrac{1}{2}r^2\theta$
3. $S = r\theta$

NOTE: YOU MAY TEAR OFF THIS FIRST PAGE AND USE (FRONT AND BACK) AS SCRATCH PAPER.

i. DO NOT START UNTIL INSTRUCTED BY A PROCTOR.

ii. THE EXAM IS ON BOTH SIDES OF EACH FOLLOWING EXAM PAGE

iii. WRITE YOUR NAME ON THE NEXT PAGE.

iv. WHEN YOU FINISH (IF BEFORE THE EXAM END TIME) PLEASE QUIETLY COLLECT YOUR THINGS AND FOLLOW PROCTOR INSTRUCTIONS IN UPLOADING YOUR EXAM WITH SUPPORTING WORK TO GRADESCOPE. ONLY WORK THAT'S SUBMITTED TO GRADESCOPE WILL BE GRADED.

Name: ____

1. Find the slant asymptote for the following rational function: (5 pts)

$$r(x) = \frac{x^3 - 5x^2 + 4x + 3}{x^2 - 3x}$$

Solution:

First, we use long division to find the quotient and the remainder

$$\begin{array}{r|rrrr}
 & x & -2 & & \\
\hline
x^2 - 3x) & x^3 & -\ 5x^2 & +\ 4x & +\ 3 \\
 & -(x^3 & -\ 3x^2) & & \\
\hline
 & & -2x^2 & +\ 4x & +\ 3 \\
 & & -\ (-2x^2 & +\ 6x) & \\
\hline
 & & & -2x & +3
\end{array}$$

Using this we can write

$$x^3 - 5x^2 + 4x + 3 = (x^2 - 3x)(x - 2) + (-2x + 3) \tag{1}$$

$$\therefore \frac{x^3 - 5x^2 + 4x + 3}{x^2 - 3x} = (x - 2) + \frac{-2x + 3}{x^2 - 3x} \tag{2}$$

Hence the end behavior of the rational function can be written as:
$r(x) \to (x - 2)$ as $x \to \infty$
Thus the slant asymptote is given by

$$\boxed{y = (x - 2)}$$

2. Answer the following for $R(x) = \dfrac{2x^2 - 4x - 16}{x^2 - 3x - 4}$ (10 pts)

   (a) Find the x-coordinate of any hole(s) in the graph of $R(x)$. If there are none write NONE.
   Solution:
   First, we factor the numerator and the denominator of the rational function:

$$R(x) = \frac{2x^2 - 4x - 16}{x^2 - 3x - 4} \tag{3}$$

$$= \frac{2\left(x^2 - 2x - 8\right)}{x^2 - 3x - 4} \tag{4}$$

$$= \frac{2(x - 4)(x + 2)}{(x - 4)(x + 1)} \tag{5}$$

   We notice that this rational function simplifies to $R'(x) = \frac{2(x+2)}{(x+1)}$ when $x \neq 4$
   Hence a hole is located at $\boxed{x = 4}$

   (b) Find the y-coordinate of any hole(s) in the graph of $R(x)$. If there are no hole(s) write NONE.
   Solution:
   We plug in $x = 3$ into this simplified function to find the y-coordinate of the hole

$$y = R'(4) = \frac{2(4 + 2)}{(4 + 1)} = \boxed{\frac{12}{5}}$$

(c) Determine the end behavior of $R(x)$.

Solution:
We notice that the Numerator and the Denominator of $R(x)$ both have degree 2. Hence as $x \to \infty$ the function approaches the ratio of the leading coefficients 2 and thus has a Horizontal Asymptote given by

$$\boxed{y = 2}$$

(d) Find all vertical asymptote(s) of $R(x)$. If there are none write NONE.

Solution:
Looking at the reduced function $R'(x) = \frac{2(x+2)}{(x+1)}$, there is Vertical Asymptote when the denominator is zero. It is given by the vertical line

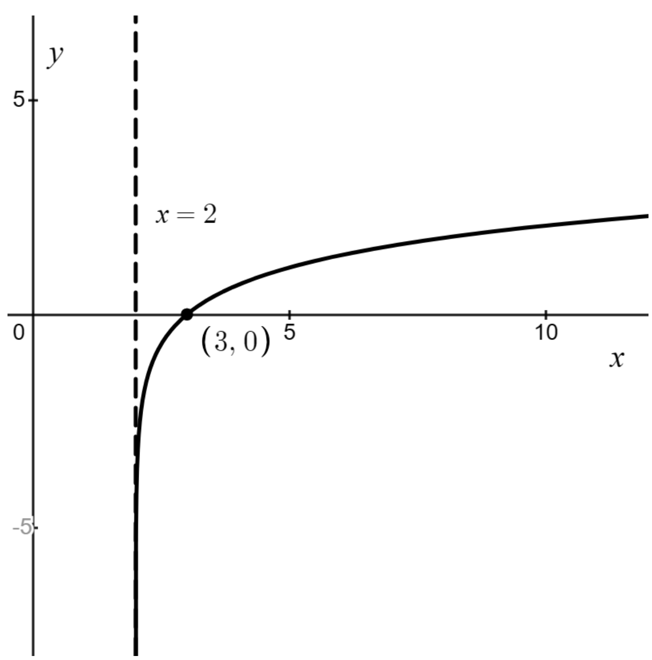
$$\boxed{x = -1}$$

3. The following parts are unrelated:

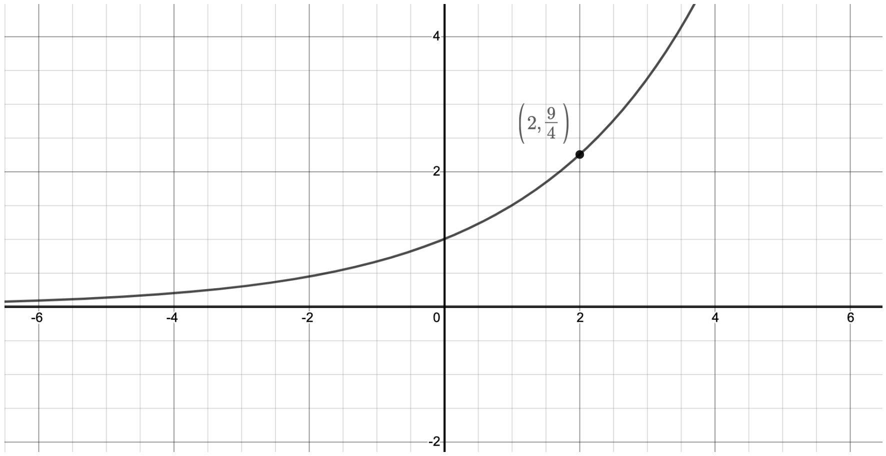
(a) Graph the following function. Please label any asymptotes and x, y-intercepts on your axes. (4 pts)

$$f(x) = \ln(x - 2)$$

Solution:



(b) Find the function of the form $y = b^x$ whose graph is given. (4 pts)



Solution:

The graph of $y = a^x$ crosses through the point $\left(2, \dfrac{9}{4}\right)$ so $\dfrac{9}{4} = a^2$. Taking the square root of both sides we get: $a = \pm\dfrac{3}{2}$. Since the graph is that of exponential growth we get: $a = \dfrac{3}{2}$. Hence the answer is:

$$\boxed{y = \left(\frac{3}{2}\right)^x}$$

4. Convert the following to exponential form (4 pts)

(a) $\log_3(9) = 2$

Solution:

$\boxed{3^2 = 9}$

(b) $\ln(1) = 0$

Solution:

$\boxed{e^0 = 1}$

5. The following are unrelated: (12 pts)

(a) Evaluate: $\log_3(27)$

Solution:

$$\begin{aligned} \log_3(27) &= \log_3(3^3) && (6)\\ &= 3\log_3(3) && (7)\\ &= \boxed{3} && (8) \end{aligned}$$

(b) Use the properties of logs to simplify the expression: $\log_2(20) - \log_2(10) - \log(1) + e^{\ln t}$

Solution:

$$\begin{aligned} \log_2(20) - \log_2(10) - \log(1) + e^{\ln t} &= \log_2\left(\frac{20}{10}\right) - 0 + t && (9)\\ &= \log_2(2) + t && (10)\\ &= \boxed{1 + t} && (11) \end{aligned}$$

(c) Express in terms of sums and differences of logarithms without exponents: $\ln\left(\frac{e^2}{z^4\sqrt{z}}\right)$

Solution:

$$\begin{aligned}
\ln\left(\frac{e^2}{z^4\sqrt{z}}\right) &= \ln\left(\frac{e^2}{z^4 z^{\frac{1}{2}}}\right) && (12)\\
&= \ln\left(\frac{e^2}{z^{\frac{9}{2}}}\right) && (13)\\
&= \ln(e^2) - \ln(z^{\frac{9}{2}}) && (14)\\
&= 2\ln(e) - \frac{9}{2}\ln(z) && (15)\\
&= \boxed{2 - \frac{9}{2}\ln(z)} && (16)
\end{aligned}$$

6. Solve the following equations. (10 pts)

(a) $2 = \log(3x+1)$

Solution:
Converting to exponential form, we obtain

$$\begin{aligned}
3x+1 &= 10^2 && (17)\\
\therefore 3x &= 10^2 - 1 && (18)\\
\therefore 3x &= 99 && (19)\\
\therefore x &= \boxed{33} && (20)
\end{aligned}$$

(b) $5^{3x+2} = 5^{x-1}$

Solution: Since exponential are one-to-one functions, we can write

$$\begin{aligned}
5^{3x+2} &= 5^{x-1} && (21)\\
\therefore 3x+2 &= x-1 && (22)\\
\therefore 2x &= -3 && (23)\\
\therefore x &= \boxed{-\frac{3}{2}} && (24)
\end{aligned}$$

7. Solve the following equations. (10 pts)

(a) $\log(x) + \log(x-4) = \log(3x)$
Solution:

$$\begin{aligned}
\log(x) + \log(x-4) &= \log(3x) && (25)\\
\log\left(x(x-4)\right) &= \log(3x) && (26)\\
x(x-4) &= 3x && (27)\\
x^2 - 4x - 3x &= 0 && (28)\\
x(x-7) &= 0 && (29)\\
x &= 0, 7 && (30)
\end{aligned}$$

However, $x = 0$ is not in the domain of $\log(x)$ and hence is an extraneous solution. Also, plugging in $x = 7$ to both sides of the equation, we find that both sides match. Hence the valid solution is

$$\boxed{x = 7}$$

(b) $2^{x+1} = 3^x$

Solution:

$$2^{x+1} = 3^x \tag{31}$$
$$\therefore \log_2(2^{x+1}) = \log_2(3^x) \tag{32}$$
$$\therefore x + 1 = x\log_2(3) \tag{33}$$
$$\therefore x(1 - \log_2(3)) = -1 \tag{34}$$
$$\therefore x = \boxed{\frac{1}{(\log_2(3) - 1)}} \tag{35}$$

Notice that in the 1st step, we could have taken logarithm of both sides w.r.t any other base of our choice (e.g 3 or e) and obtained a solution which is equivalent to the one presented above.

8. A certain type of beetle arrives in Colorado for the first time. The beetle population within the state is expected to double every 8 years. Suppose the initial population is 10 beetles. (6 pts)

(a) Find an exponential model of the form $N(t) = N_\circ 2^{\frac{t}{a}}$ that models the population of beetles where $N(t)$ is the number of beetles and t is time in years

Solution:
Since the initial population is 10 and the doubling time is 8

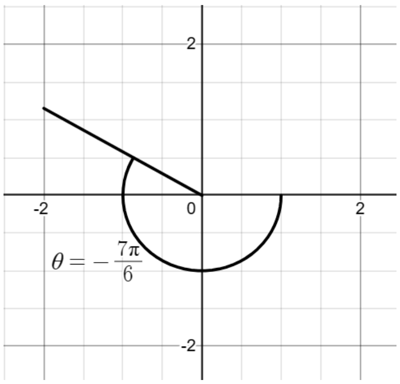
$$\boxed{N(t) = 10 * 2^{\frac{t}{8}}}$$

(b) What will the beetle population be after 16 years? As usual, give your answer in exact form. Do not attempt to approximate with a decimal value

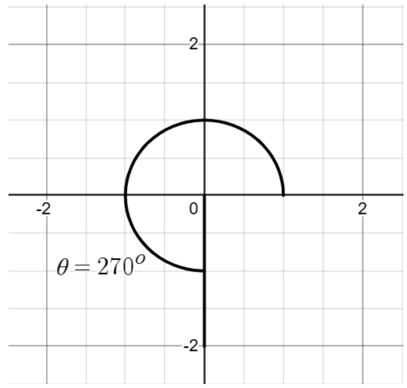
Solution:
Since the population doubles every 8 years, after 16 years it will become
$10 * 2 * 2 = \boxed{40}$ beetles.

9. Sketch each angle, θ, in standard position on the x,y-axes. Give a separate graph for each.

(a) $\theta = -\dfrac{7\pi}{6}$ (3 pts)



(b) $\theta = 270^o$ (3 pts)



10. Evaluate the following: (15 pts)

Solution:
The following can be obtained from a Unit Circle

(a) $\sin(0)$

$$\boxed{\sin(0) = 0}$$

(b) $\tan(-120^o)$

$$\boxed{\tan(-120^o) = \sqrt{3}}$$

(c) $\sin\left(-\dfrac{5\pi}{4}\right)$

$$\boxed{\sin\left(-\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}}$$

(d) $\cos\left(\dfrac{2\pi}{3}\right)$

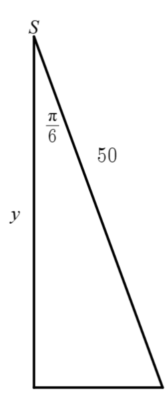
$$\boxed{\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}}$$

(e) $\csc\left(\dfrac{\pi}{3}\right)$

$$\boxed{\csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}}$$

11. A squirrel clings to the trunk of a tree, and she sees a peanut on the flat ground some distance away. A straight line can be drawn between the squirrel and peanut. If the straight-line distance between the peanut and the squirrel is 50 ft, and the angle between that straight line and the tree trunk is $\dfrac{\pi}{6}$ radians, how far down the tree and across the ground must the squirrel travel to reach the peanut? *Give your answer with appropriate units.* (6 pts)

Solution:
Say,the position of the squirrel on the top the tree is marked by an S in the following picture



From the triangle above, we see that distance down the tree is obtained by:

$$\cos\left(\frac{\pi}{6}\right) = \frac{y}{50} \tag{36}$$
$$\therefore y = 50\cos\left(\frac{\pi}{6}\right) \tag{37}$$
$$= 25\sqrt{3} \tag{38}$$

Similarly, the distance along the ground is given by

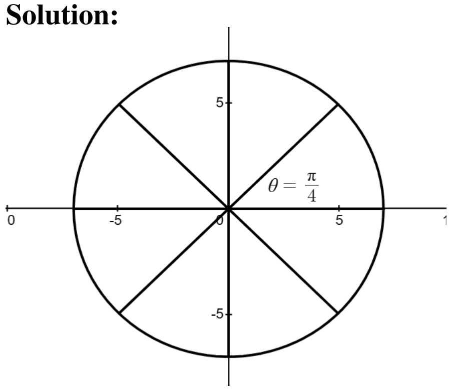
$$\sin\left(\frac{\pi}{6}\right) = \frac{x}{50} \tag{39}$$
$$\therefore x = 50\sin\left(\frac{\pi}{6}\right) \tag{40}$$
$$= 25 \tag{41}$$

Hence the total distance is given by

$$D = x + y \tag{42}$$
$$= \boxed{25\left(1+\sqrt{3}\right)\text{ ft}} \tag{43}$$

12. A pizza, whose surface is in the shape of a circle with a radius of 7 inches, is cut into eight equally sized slices such that each slice forms a circular sector. What is the area of the surface of each slice? (4 pts)

Solution:



As seen from the above picture, each slice of the pizza subtends an angle $\theta = \frac{1}{8}(2\pi) = \frac{\pi}{4}$
Hence the area of each sector is given by

$$A = \frac{1}{2}r^2\theta \tag{44}$$
$$= \frac{1}{2}7^2\left(\frac{\pi}{4}\right) \tag{45}$$
$$= \boxed{\frac{49\pi}{8}\ in^2} \tag{46}$$

Alternative Solution:
The total area of the pizza is given by

$$A = \pi 7^2 \tag{47}$$
$$= 49\pi \tag{48}$$

Hence area of each slice is given by $A = \frac{1}{8}49\pi = \boxed{\frac{49\pi}{8}\ in^2}$

13. Knowing that $\cos^2\theta + \sin^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$ write $\tan\theta$ in terms of $\sin\theta$ if θ is in Quadrant I (4 pts)
Solution:

$$\tan^2\theta + 1 = \sec^2\theta \tag{49}$$

$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta} \tag{50}$$

$$\tan^2\theta + 1 = \frac{1}{1-\sin^2\theta} \tag{51}$$

$$\tan^2\theta = \frac{1}{1-\sin^2\theta} - 1 \tag{52}$$

$$\tan^2\theta = \frac{\sin^2\theta}{1-\sin^2\theta} \tag{53}$$

$$\tan\theta = \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \tag{54}$$

In Quadrant I, $\tan\theta > 0$. Hence we take the positive square root and obtain

$$\boxed{\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}}$$