INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, electronic devices (such as calculator or other unauthorized electronic resources) are not permitted. **Give all answers in exact form.**

Potentially useful formulas:

$$\log_b(B) = \frac{\log_a(B)}{\log_a(b)} \text{ for } a > 0,\ a \neq 1.$$

$$A = \frac{1}{2}r^2\theta$$

$$S = r\theta$$

1. For $R(x) = \dfrac{x^2 - 4}{x^3 + 10x^2 + 16x}$ (9 pts)

(a) Find the location (x, y -coordinates) of any hole(s). If there are none state NONE.

Solution:

Factor the numerator and denominator to see if any terms cancel:

$$\frac{x^2 - 4}{x^3 + 10x^2 + 16x} = \frac{(x-2)}{x(x+8)}$$

Because $(x + 2)$ is both in the numerator and denominator, they can cancel and the hole is at $x = -2$.

To find the y-coordinate of the hole, we plug in the x value into the simplified version of $R(x)$:

$$\text{plugging in } x = -2\text{: } \frac{-2-2}{-2(-2+8)} = \frac{-4}{-12} = \frac{1}{3}$$

Therefore, the hole is at $\boxed{\left(-2, \frac{1}{3}\right)}$.

Note: Because the rational function has a hole in it, we use the simplified version of the function to find vertical and horizontal asymptotes.

(b) Find any horizontal or slant asymptote(s). If there are none state NONE.

Solution:

Horizontal asymptotes occur when the power of the numerator is the same or smaller than the power of the denominator. Because the power of the numerator of the simplified function is 1 and the power of the denominator of the simplified function is 2, we find the horizontal asymptote by dividing the term of the largest power in the numerator by the term of the largest power in the denominator:

$R(x) \approx \dfrac{x}{x^2} = \dfrac{1}{x} \to 0$ as $x \to \pm\infty$.

Therefore, the horizontal asymptote is at $\boxed{y = 0}$, and there are no slant asymptotes.

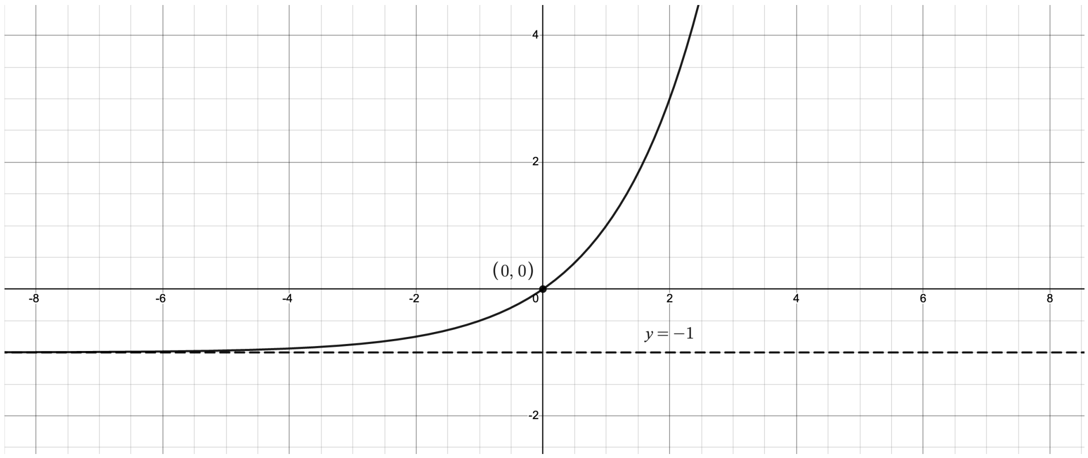
(c) Find any vertical asymptote(s). If there are none state NONE.

Solution:

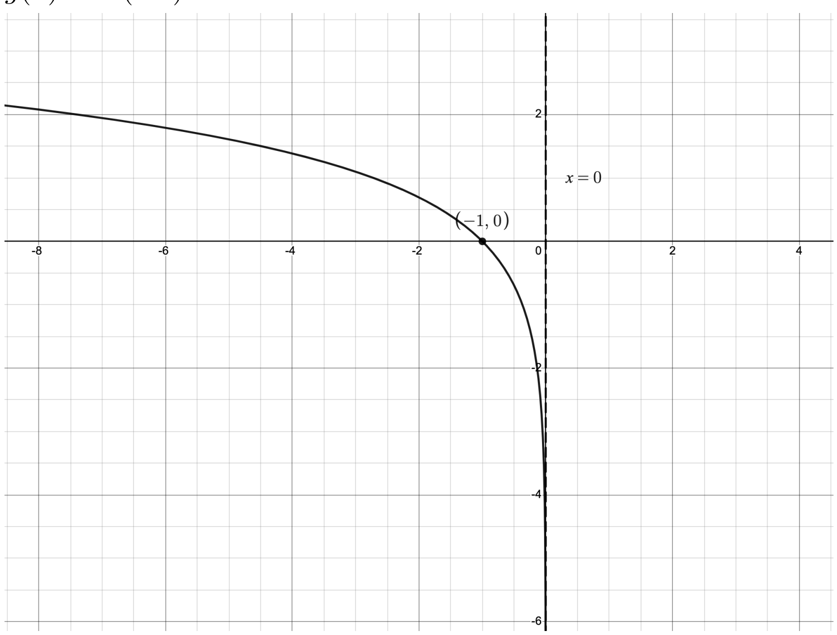
Vertical asymptotes occur wherever the denominator of the simplified function equals zero. Thus, we set the denominator equal to zero and solve $x(x+8)=0$ resulting in two vertical asymptotes: $\boxed{x=0}$ and $\boxed{x=-8}$.

2. Sketch the following graphs: Be sure to label any asymptotes and intercepts for each graph. (8 pts)

(a) $f(x) = 2^x - 1$



(b) $g(x) = \ln(-x)$



3. (a) Simplify (rewrite without logs): $\ln(1) - \log_2(16) + \log_8(4) + \log_8(2) + 2^{3\log_2(4)}$ (4 pts)

Solution:

$$\begin{aligned} \ln(1) - \log_2(16) + \log_8(4) + \log_8(2) + 2^{3\log_2(4)} &= 0 - 4 + \log_8(8) + 2^{\log_2(4^3)} \\ &= 0 - 4 + 1 + 4^3 \\ &= \boxed{61} \end{aligned}$$

(b) Rewrite as the sum/difference of logarithms without exponents: $\ln\left(\frac{e^3a^2}{\sqrt{b}}\right)$ (4 pts)

Solution:

$$\begin{aligned}\ln\left(\frac{e^3a^2}{\sqrt{b}}\right) &= \ln\left(e^3a^2\right) - \ln\left(\sqrt{b}\right)\\ &= \ln\left(e^3\right) + \ln\left(a^2\right) - \ln\left(\sqrt{b}\right)\\ &= \boxed{3 + 2\ln(a) - \frac{1}{2}\ln(b)}\end{aligned}$$

4. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit).

(a) $\log(x+2) = \log(3-2x)$ (4 pts)

Solution:

We'll start by applying the one-to-one property for logarithms, that is, if $\log(x_1) = \log(x_2)$ then $x_1 = x_2$.

$$\begin{aligned}\log(x+2) &= \log(3-2x)\\ x+2 &= 3-2x\\ 3x &= 1\\ x &= \boxed{\frac{1}{3}}\end{aligned}$$

Checking $x = \frac{1}{3}$ we see it is indeed a solution of the original equation.

(b) $3^{x^2-1} = 9$ (4 pts)

Solution:

$$\begin{aligned}3^{x^2-1} &= 3^2\\ x^2-1 &= 2\\ x^2 &= 3\\ x &= \boxed{\pm\sqrt{3}}\end{aligned}$$

5. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit).

(a) $\log(2x) = \log\left(\frac{1}{3}x - 2\right) + \log(3)$ (4 pts)

Solution:

$$\begin{aligned} \log(2x) &= \log\left(\frac{1}{3}x - 2\right) + \log(3) \\ \log(2x) &= \log\left(\left(\frac{1}{3}x - 2\right)3\right) \\ \log(2x) &= \log(x - 6) \\ 2x &= x - 6 \\ x &= -6 \end{aligned}$$

However, checking this potential solution in the original equation we find there is $\boxed{\text{no solution}}$.

(b) $\log_2(5)x = x - 1$ (4 pts)

Solution:

Since $\log_2(5)$ is a constant, we can make our equation look more familiar by letting $c = \log_2(5)$.

$$\begin{aligned} \log_2(5)x &= x - 1 \\ cx &= x - 1 \\ cx - x &= -1 \\ x(c - 1) &= -1 \\ x &= -\frac{1}{c - 1} \\ x &= \boxed{-\frac{1}{\log_2(5) - 1}} \end{aligned}$$

6. Simplify the expression: $(e^x - e^{-x})(e^x + e^{-x}) - (e^x)^2$ (4 pts)

Solution:

$$\begin{aligned} (e^x - e^{-x})(e^x + e^{-x}) - (e^x)^2 &= e^{2x} - 1 + 1 - e^{-2x} - e^{2x} \\ &= \boxed{-e^{-2x}} \end{aligned}$$

7. The temperature of a cup of coffee in a room is modeled to cool according to the temperature model $T(t) = 110e^{-0.08t} + 75$ where T is the temperature of the coffee in degrees Fahrenheit and t is the time in minutes.

   (a) What is the initial temperature of the coffee? (3 pts)

   Solution:

$$\begin{aligned} T_\circ &= T(0) \\ &= 110e^{-0.08(0)} + 75 \\ &= 110e^0 + 75 \\ &= 110(1) + 75 \\ &= \boxed{185^\circ \text{ F}} \end{aligned}$$

   (b) What is the temperature of the coffee after 100 minutes (do not attempt to approximate the value with decimals)? (4 pts)

   Solution:

$$\begin{aligned} T(100) &= 110e^{-0.08(100)} + 75 \\ &= \boxed{110e^{-8} + 75} \end{aligned}$$

   (c) According to the model, what is the temperature of the room? (2 pts)

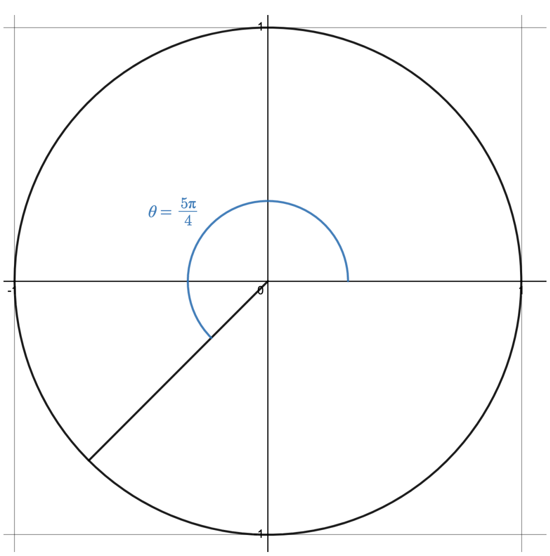
   Solution:

   The coffee cools, approaching the temperature of the room. As $t \to \infty$ then $T \to 75$ since $110e^{-0.08t} = \dfrac{100}{e^{0.08t}} \to 0$. We can also find this by interpreting the $+\ 75$ in the model as a vertical shift of the exponential decay graph, so the coffee cools to 75 degrees F (the room temperature). So the room temperature is $\boxed{75^\circ \text{ F}}$.

8. Sketch each angle in standard position on the unit circle.

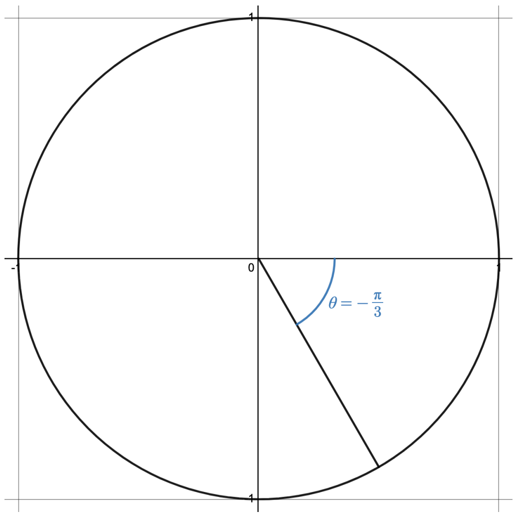
   (a) $\theta = \dfrac{5\pi}{4}$ (2 pts)

   Solution:



(b) $\theta = -\dfrac{\pi}{3}$ (2 pts)

Solution:



9. The following are unrelated.

(a) The point, (x, y), lies on the unit circle $x^2 + y^2 = 1$ in quadrant II. If $y = \dfrac{\sqrt{3}}{5}$ find x. (4 pts)

Solution:

Since $x^2 + y^2 = 1$ and $y = \dfrac{\sqrt{3}}{5}$ then we can find x by substituting in $y = \dfrac{\sqrt{3}}{5}$ to get $x^2 + \left(\dfrac{\sqrt{3}}{5}\right)^2 = 1$

$$\begin{aligned} x^2 + \left(\frac{\sqrt{3}}{5}\right)^2 &= 1 \\ x^2 + \frac{3}{25} &= 1 \\ x^2 &= 1 - \frac{3}{25} \\ x^2 &= \frac{22}{25} \\ x &= \pm\sqrt{\frac{22}{25}} \\ x &= \pm\frac{\sqrt{22}}{5} \end{aligned}$$

Since θ is in quadrant II then the x-coordinate is negative and so the answer is: $\boxed{x = -\dfrac{\sqrt{22}}{5}}$.

(b) For an angle θ in standard position, suppose we know $\cos(\theta) < 0$ and $\sin(\theta) < 0$. What quadrant does the terminal side of θ lie? (3 pts)

Solution:

Since $\cos(\theta) < 0$ is only true in quadrants II and III, and $\sin(\theta) < 0$ is only true in quadrants III and IV, θ must be in $\boxed{\text{Quadrant III.}}$

(c) Find two angles that are co-terminal with $\theta = 160°$. (3 pts)

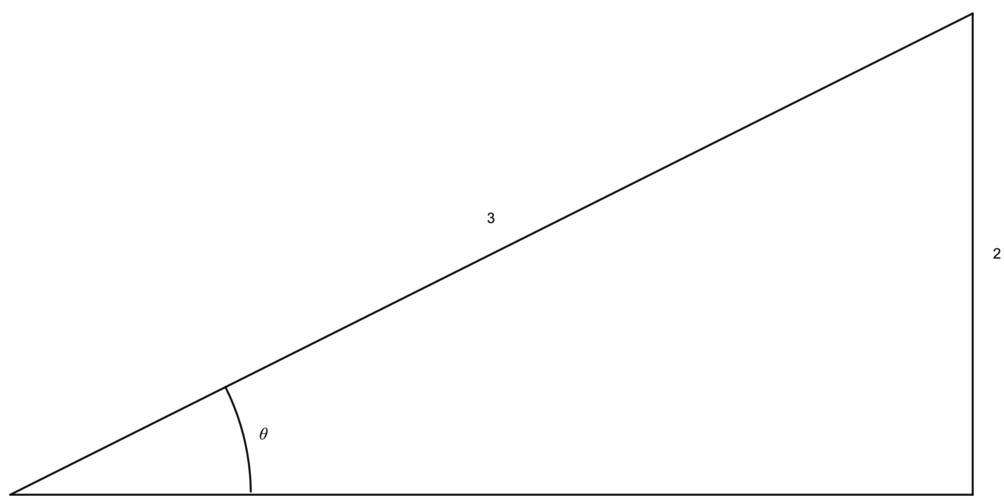
Solution:

One is found by adding $360°$ to get $160° + 360° = 520°$. We can add $360°$ again to get $520° + 360° = 880°$. Two possible answers are: $\boxed{880°}$ and $\boxed{520°}$. Note that we could have subtracted $360°$ as well. There are an infinite number of possible answers.

10. Answer the following for $\sin(\theta) = \frac{2}{3}$ where θ lies in the interval $\left[0, \frac{\pi}{2}\right]$.

(a) Sketch a triangle that has acute angle θ (3 pts).

Solution:



(b) Find $\tan\theta$ (3 pts)

Solution:

The length of the missing side can be found by using the Pythagorean identity: $a^2 + b^2 = c^2$. Here, $c = 3$ and $b = 2$ so $a = \sqrt{9-4} = \sqrt{5}$.

So $\tan\theta = \dfrac{opp}{adj} = \boxed{\dfrac{2}{\sqrt{5}}}$. Note $\tan\theta$ is positive for θ in $\left[0, \frac{\pi}{2}\right]$.

(c) Find $\csc\theta$ (3 pts)

Solution:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{2}{3}} = \boxed{\frac{3}{2}}.$$

11. Find the exact value of each of the following. If a value does not exist write DNE.

(a) $\cos\left(\frac{3\pi}{4}\right)$ (3 pts)

Solution:

$$\cos\left(\frac{3\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

(b) $\sin(120°)$ (3 pts)

Solution:

$$\sin(120°) = \boxed{\frac{\sqrt{3}}{2}}$$

(c) $\tan(0)$ (3 pts)

Solution:

$$\tan(0) = \boxed{0}$$

(d) $\cot\left(\frac{\pi}{4}\right)$ (3 pts)

Solution:

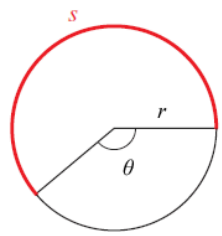
$$\cot\left(\frac{\pi}{4}\right) = \boxed{1}$$

(e) $\sec\left(-\frac{\pi}{3}\right)$ (3 pts)

Solution:

$$\sec\left(-\frac{\pi}{3}\right) = \boxed{2}$$

12. Given that $r = 3$ and $\theta = 140°$. Find the length s of the circular arc as shown in the image. (4 pts)



Solution:

The angle for sector we're interested in is found by subtracting $140°$ from $360°$. $360° - 140° = 220°$. Converting to radians, we get: $220° \dfrac{\pi}{180°} = \dfrac{11\pi}{9}$. So, using $s = r\theta$ we get $s = 3 \cdot \dfrac{11\pi}{9} = \boxed{\dfrac{11\pi}{3}}$.

13. The angle of elevation to the top of a very tall building is found to be $30°$ from the ground at a distance of 0.8 mile from the base of the building. Using this information, find the height of the building (do not attempt to approximate your answer as a decimal). (4 pts)

Solution:

Letting h represent the height of the building and setting up a right triangle we can use tangent to write $\tan(30°) = \dfrac{h}{0.8}$. So $h = 0.8 \tan(30°) = 0.8 \dfrac{1}{\sqrt{3}} = \dfrac{4}{5} \dfrac{1}{\sqrt{3}} = \boxed{\dfrac{4}{5\sqrt{3}} \text{ miles}}$.