INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, electronic devices (such as calculator or other unauthorized electronic resources) are not permitted. **Give all answers in exact form.**

1. The following are unrelated. Simplify answers and leave without negative exponents. (28 pts)

   (a) Simplify: $5x(x-x^3)-(-3x^2-1)^2$

   Solution

$$5x(x-x^3)-(-3x^2-1)^2 = 5x^2-5x^4-(-3x^2-1)(-3x^2-1) \quad (1)$$
$$= 5x^2-5x^4-(9x^4+3x^2+3x^2+1) \quad (2)$$
$$= 5x^2-5x^4-9x^4-6x^2-1 \quad (3)$$
$$= -14x^4-x^2-1 \quad (4)$$

   (b) Simplify: $(-xy^3)^2\left(\frac{x^{-5}}{y^{-2}}\right)+3\frac{y^8}{x^3}$

   Solution

$$(-xy^3)^2\left(\frac{x^{-5}}{y^{-2}}\right)+3\frac{y^8}{x^3} = x^2y^6\left(\frac{y^2}{x^5}\right)+3\frac{y^8}{x^3} \quad (5)$$
$$= \frac{y^8}{x^3}+3\frac{y^8}{x^3} \quad (6)$$
$$= 4\frac{y^8}{x^3} \quad (7)$$

(c) Add: $\dfrac{x}{x^2-1}+\dfrac{1}{x^2+x}$

Solution

$$\frac{x}{x^2-1}+\frac{1}{x^2+x} = \frac{x}{(x-1)(x+1)}+\frac{1}{x(x+1)} \tag{8}$$
$$= \frac{x^2}{(x-1)(x+1)x}+\frac{x-1}{x(x+1)(x-1)} \tag{9}$$
$$= \frac{x^2+x-1}{x(x-1)(x+1)} \tag{10}$$

Note that x^2+x-1 does not factor in a way that leads to further simplification.

(d) Multiply: $(24i-8)(4i+1)$. Give answer in $a+bi$ form.

Solution

$$(24i-8)(4i+1) = 96i^2+24i-32i-8 \tag{11}$$
$$= 96(-1)-8i-8 \tag{12}$$
$$= -104-8i \tag{13}$$

(e) Multiply: $\left(x^{1/2}-y^{1/3}\right)\left(x^{1/2}+y^{1/3}\right)$

Solution

$$\left(x^{1/2}-y^{1/3}\right)\left(x^{1/2}+y^{1/3}\right) = x^{1/2}x^{1/2}+x^{1/2}y^{1/3}-y^{1/3}x^{1/2}-y^{1/3}y^{1/3} \tag{14}$$
$$= x-y^{2/3} \tag{15}$$

(f) Simplify: $\dfrac{\frac{1}{x-2}+1}{\frac{1}{x-2}}$

Solution

$$\frac{\frac{1}{x-2}+1}{\frac{1}{x-2}} = \frac{\frac{1}{x-2}+\frac{x-2}{x-2}}{\frac{1}{x-2}} \tag{16}$$
$$= \frac{\frac{x-1}{x-2}}{\frac{1}{x-2}} \tag{17}$$
$$= x-1 \tag{18}$$

(g) $3^{4\log_3(x)} + \log_8(4) + \log_8(2) - \ln\left(e^2\right) + \log_2(32)$ (Your answer should have no logarithms)

Solution

$$\begin{aligned}
3^{4\log_3(x)} + \log_8(4) + \log_8(2) - \ln\left(e^2\right) + \log_2(32) &= 3^{\log_3\left(x^4\right)} + \log_8(4\cdot 2) - 2 + 5 && (19)\\
&= x^4 + \log_8(8) + 3 && (20)\\
&= x^4 + 1 + 3 && (21)\\
&= x^4 + 4 && (22)
\end{aligned}$$

2. Solve the following equations for x: (16 pts)

(a) $\frac{1}{3}x - \frac{3}{4} = \frac{1}{2}x$

Solution

$$\begin{aligned}
\frac{1}{3}x - \frac{3}{4} &= \frac{1}{2}x && (23)\\
12\left(\frac{1}{3}x - \frac{3}{4}\right) &= 12\left(\frac{1}{2}x\right) && (24)\\
4x - 9 &= 6x && (25)\\
-9 &= 2x && (26)\\
x &= -\frac{9}{2} && (27)
\end{aligned}$$

(b) $\sqrt{6} - 3x^2 = x^2$

Solution

$$\begin{aligned}
\sqrt{6} - 3x^2 &= x^2 && (28)\\
\sqrt{6} &= 4x^2 && (29)\\
\frac{\sqrt{6}}{4} &= x^2 && (30)\\
\pm\sqrt{\frac{\sqrt{6}}{4}} &= x && (31)\\
x &= \pm\frac{\sqrt[4]{6}}{2} && (32)
\end{aligned}$$

Leading to two possible answers: $x = \pm\frac{\sqrt[4]{6}}{2}$. Checking both answers in the original equation we see that both solve the original equation.

(c) $\log_2(1-x) - \log_2(x) = 2$

Solution

$$\log_2(1-x) - \log_2(x) = 2 \tag{33}$$
$$\log_2\left(\frac{1-x}{x}\right) = 2 \tag{34}$$
$$\frac{1-x}{x} = 2^2 \tag{35}$$
$$1 - x = 4x \tag{36}$$
$$1 = 5x \tag{37}$$
$$x = \frac{1}{5} \tag{38}$$

Checking $x = \frac{1}{5}$ in the original equation we see that it solves the original equation.

(d) $6 = 2e^{-x-1}$

Solution

$$6 = 2e^{-x-1} \tag{39}$$
$$3 = e^{-x-1} \tag{40}$$
$$\ln(3) = -x - 1 \tag{41}$$
$$\ln(3) + 1 = -x \tag{42}$$
$$-\ln(3) - 1 = x \tag{43}$$

3. For $g(x) = \dfrac{x^3 + 5x^2 + 6x}{x^2 - 9}$ answer the following (16 pts):

(a) Find the domain of $g(x)$. Give your answer in interval notation.

Solution

The domain is all real numbers except where $x^2 - 9 = 0$.

$$x^2 - 9 = 0 \tag{44}$$
$$(x-3)(x+3) = 0 \tag{45}$$

So the domain is all real numbers except $x = -3, 3$. In interval notation: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

(b) Find the x, y-coordinates for any hole(s). If there are none write NONE.

Solution

Factoring the numerator and denominator: $g(x) = \dfrac{x^3 + 5x^2 + 6x}{x^2 - 9} = \dfrac{x(x+3)(x+2)}{(x-3)(x+3)} = \dfrac{x(x+2)}{x-3}$ if $x \neq -3$. So there is a hole at $x = -3$. We substitute $x = -3$ into the reduced fraction and get the y-coordinate: $y = \dfrac{-3(-3+2)}{-3-3} = \dfrac{3}{-6} = -\dfrac{1}{2}$.

So the coordinates of the hole are: $\left(-3, -\dfrac{1}{2}\right)$.

(c) Find any horizontal or slant asymptotes. If there are none write NONE.

Solution

Since the degree in the numerator is exactly one more than the denominator then there is a slant asymptote. To find the slant asymptote we utilize long division:

$$\begin{array}{r l}
 & x + \quad 5 \\ \cline{2-2}
x - 3 \,\big) & x^2 + \; 2x + 0 \\
 & -(x^2 - 3x) \\ \cline{2-2}
 & \qquad\quad 5x + 0
\end{array}$$

We can stop the long division as soon as we have the slant asymptote and find the slant asymptote is $y = x + 5$.

(d) Find any vertical asymptotes. If there are none write NONE.

Solution

There is a vertical asymptote when the simplified fraction results in division by zero. Setting $x - 3 = 0$ we get one vertical asymptote of $x = 3$.

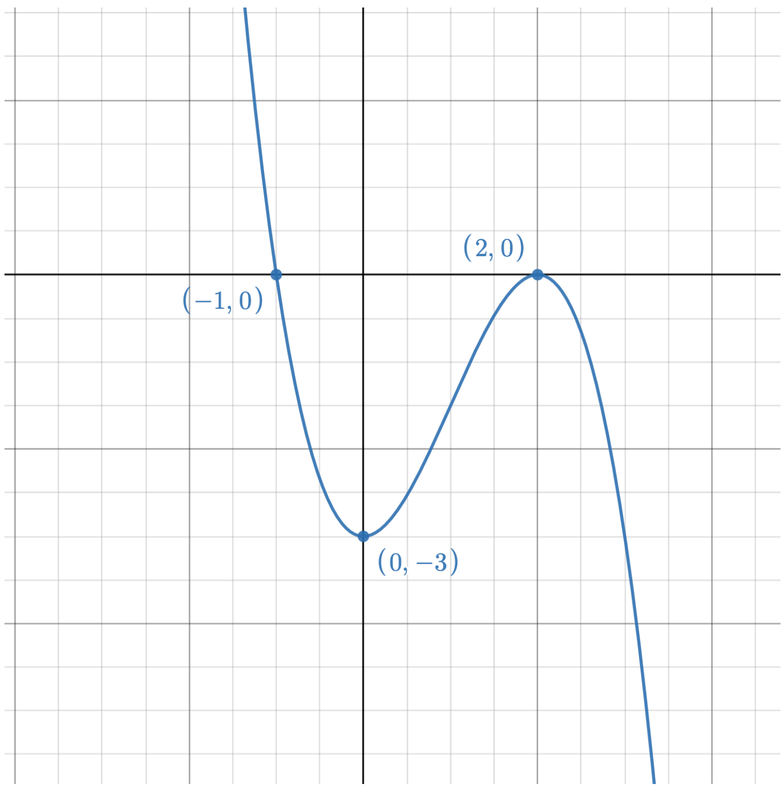
4. Answer the following for the polynomial $P(x)$. (8 pts)

(a) Sketch the graph of $y = P(x)$ with the following properties:

i. $y \to \infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.

ii. The graph crosses the x-axis at $x = -1$ and touches, but does not cross, the x-axis at $x = 2$. The graph has y-intercept of $(0, -3)$. The graph has no other intercepts.

Solution



(b) Write down a polynomial that satisfies the given information.

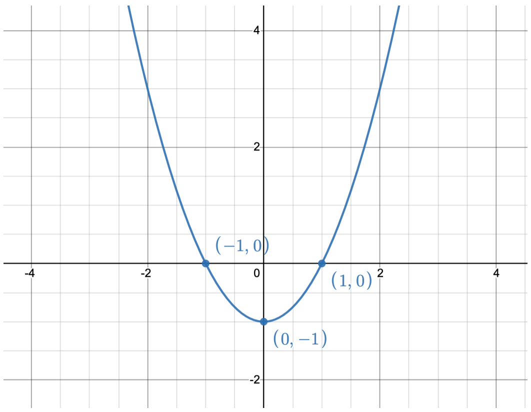
Solution

Starting with the x-intercepts we get: $P(x) = (x-2)^2(x+1)$. However this polynomial has the wrong end behavior and a y-intercept of $(0, 4)$. Multiplying the polynomial by $-\frac{3}{4}$ results in the correct end behavior and y-intercept. The answer then is: $P(x) = -\frac{3}{4}(x-2)^2(x+1)$. Note there are other possible answers such as: $P(x) = -\frac{3}{16}(x-2)^4(x+1)^3$.

5. Sketch the graph of the following functions. Label all intercepts and asymptotes as appropriate. (13 pts)

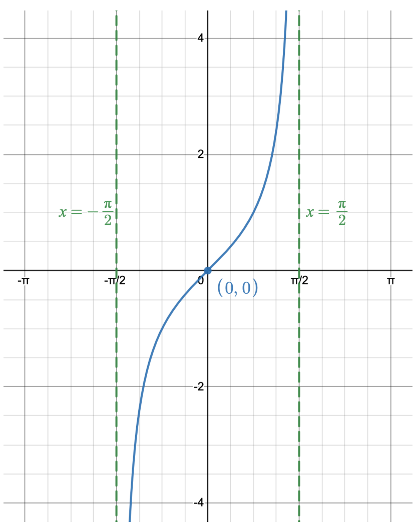
(a) $f(x) = x^2 - 1$

Solution



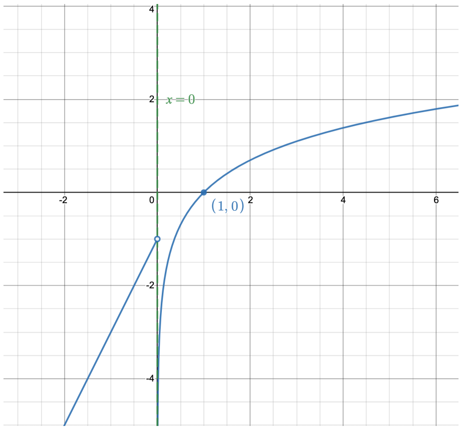
(b) $h(x) = \tan(x)$ on the restricted domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution



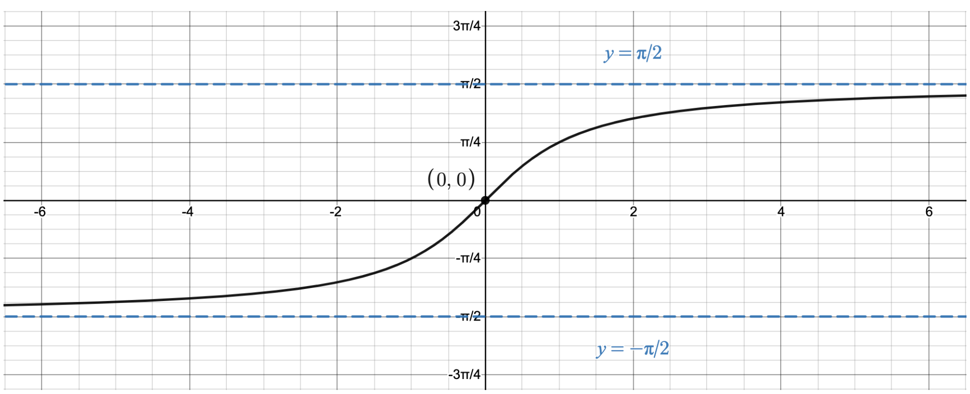
(c) $q(x) = \begin{cases} 2x - 1 & \text{if} \quad x < 0 \\ \ln x & \text{if} \quad x > 0 \end{cases}$

Solution



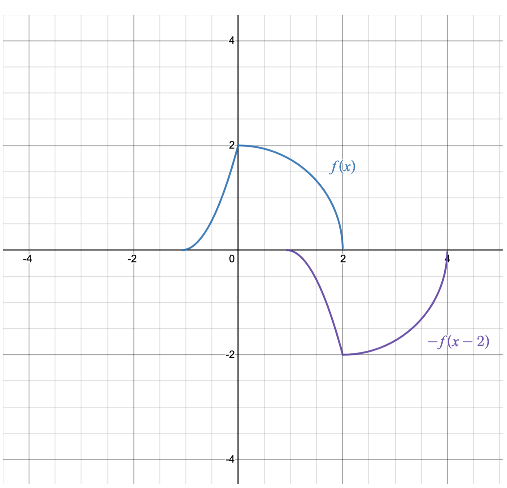
(d) $k(x) = \tan^{-1}(x)$

Solution



6. For the given graph of $f(x)$ in blue; sketch the shape of the graph $-f(x-2)$ on the **same graph**. (4 pts)

Solution



7. You are standing 48 meters from the base of a building. You estimate that the angle of elevation to the top of the 82nd floor (the observatory) is $77°$. If the total height of the building is another 109 meters above the 82nd floor, what is the height of the building? (5 pts)

Solution

Let h represent the height of the 82nd floor. The height to the 82nd floor is found with the relationship $\tan(77°) = \frac{h}{48}$ giving us $h = 48\tan(77°)$. Since the top of the building is another 109 meters above the 82nd floor then the total height of the building is: $48\tan(77°) + 109$ meters.

8. Given $f(x) = \sqrt{x}$ and $g(x) = x^4 - 2x^2 - 1$ answer the following: (7 pts)

(a) Find $(g \circ f)(x)$ and find the domain.

Solution

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) && (46)\\
&= g\left(\sqrt{x}\right) && (47)\\
&= \left(\sqrt{x}\right)^4 - 2\left(\sqrt{x}\right)^2 - 1 && (48)\\
&= x^2 - 2x - 1 && (49)
\end{aligned}$$

The domain of $(g \circ f)(x)$ is the intersection of the domain of $x^2 - 2x - 1$ and $f(x) = \sqrt{x}$. So the domain of the composition is $[0, \infty)$.

(b) Is $g(x)$ odd, even, or neither? Justify your answer for credit.

Solution

$$\begin{aligned}
g(-x) &= (-x)^4 - 2(-x)^2 - 1 && (50)\\
&= x^4 - 2x^2 - 1 && (51)\\
&= g(x) && (52)
\end{aligned}$$

Since $g(-x) = g(x)$ then $g(x)$ is even.

9. Find the exact value: (15 pts)

(a) $\sin\left(\frac{5\pi}{4}\right)$

Solution

$$\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

(b) $\cos\left(-\frac{\pi}{6}\right)$

Solution

$$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

(c) $\arctan(-1)$

Solution

$$\arctan(-1) = -\frac{\pi}{4}$$

(d) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$

Solution

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

(e) $\cos\left(\cos^{-1}(1)\right)$

Solution

$$\cos\left(\cos^{-1}(1)\right) = \cos(0) = 1$$

10. For a specific angle θ suppose we know that $\sin\theta > 0$ and θ lies in the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. What quadrant does θ lie in? (4 pts)

Solution

$\sin\theta > 0$ in quadrants I and II and $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ specifies angles in quadrants II and III so the only quadrant that satisfies both pieces of information is Quadrant II.

11. Verify the identity: $\dfrac{1+\sec\theta}{\tan\theta+\sin\theta} = \csc\theta$. (5 pts)

Solution

Starting with the left hand side we convert each term to be in terms of $\sin\theta$ and $\cos\theta$ and simplify the complex fraction:

$$\frac{1+\sec\theta}{\tan\theta+\sin\theta} = \frac{1+\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}+\sin\theta} \tag{53}$$

$$= \frac{\frac{\cos\theta}{\cos\theta}+\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}+\sin\theta\frac{\cos\theta}{\cos\theta}} \tag{54}$$

$$= \frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+\sin\theta\cos\theta}{\cos\theta}} \tag{55}$$

$$= \frac{\cos\theta+1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta+\sin\theta\cos\theta} \tag{56}$$

$$= \frac{\cos\theta+1}{\sin\theta+\sin\theta\cos\theta} \tag{57}$$

$$= \frac{\cos\theta+1}{\sin\theta(1+\cos\theta)} \tag{58}$$

$$= \frac{1}{\sin\theta} \tag{59}$$

$$= \csc\theta_{//} \tag{60}$$

12. Suppose we know that $\sec\theta = -\dfrac{9}{4}$ for θ in quadrant II. Find $\sin(2\theta)$. (4 pts)

Solution

$\sin(2\theta) = 2\sin\theta\cos\theta$ so we need to find $\sin\theta$ and $\cos\theta$.

To find $\cos\theta$ we see that: $\sec\theta = -\dfrac{9}{4}$ so $\cos\theta = -\dfrac{4}{9}$.

To find $\sin\theta$ we set up a reference triangle and use the Pythagorean theorem to get $\sin\theta = \dfrac{\sqrt{65}}{9}$.

The answer is then: $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\left(\dfrac{\sqrt{65}}{9}\right)\cdot\left(-\dfrac{4}{9}\right) = -\dfrac{8\sqrt{65}}{81}$.

13. Find all solutions to the following equations: (8 pts)

(a) $\sin(3\theta) = \frac{1}{2}$

Solution

$\sin(3\theta) = \frac{1}{2}$ when $3\theta = \frac{\pi}{6} + 2n\pi$ and $3\theta = \frac{5\pi}{6} + 2n\pi$ where n is any integer. Dividing both sides by 3 we get solutions: $\theta = \frac{\pi}{18} + \frac{2n\pi}{3}$ and $\theta = \frac{5\pi}{18} + \frac{2n\pi}{3}$.

(b) $\frac{\sqrt{3}}{2}\tan\theta + \tan\theta\cos\theta = 0$

Solution

$$\frac{\sqrt{3}}{2}\tan\theta + \tan\theta\cos\theta = 0 \tag{61}$$

$$\tan\theta\left(\frac{\sqrt{3}}{2} + \cos\theta\right) = 0 \tag{62}$$

We get two equations to solve: $\tan\theta = 0$ and $\cos\theta = -\frac{\sqrt{3}}{2}$. The first equation, $\tan\theta = 0$, has solutions $\theta = 2n\pi$ and $\theta = \pi + 2n\pi$ where n is any integer (we can condense these solutions into $\theta = n\pi$). The second equation, $\cos\theta = -\frac{\sqrt{3}}{2}$, has solutions $\theta = \frac{5\pi}{6} + 2n\pi$ and $\theta = \frac{7\pi}{6} + 2n\pi$. So the solutions of the original equation are: $\theta = n\pi$, $\theta = \frac{5\pi}{6} + 2n\pi$, and $\theta = \frac{7\pi}{6} + 2n\pi$ for n any integer.

14. Find the exact value for each: (8 pts)

(a) $3\cos^2(-13°) + 3\sin^2(-13°)$

Solution

$3\cos^2(-13°) + 3\sin^2(-13°) = 3\left(\cos^2(-13°) + \sin^2(-13°)\right) = 3(1) = 3$

(b) $\cos\left(\frac{\pi}{12}\right)$

Solution

Using a half angle formula and noting $\cos\left(\frac{\pi}{12}\right) > 0$ we get $\cos\left(\frac{\frac{\pi}{6}}{2}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{6}\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$. Note that we could also use the difference of two angles formula here with $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$.

15. For $f(x) = 3\cos(2x)$ (9 pts)

(a) Identify the amplitude.

Solution

3

(b) Identify the period.

Solution

π

(c) Identify the phase shift.

Solution

0

(d) Sketch one cycle of the graph of $f(x)$. Be sure to label relevant values on x and y axes.