INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, electronic devices (such as calculator or other unauthorized electronic resources) are not permitted. **Give all answers in exact form.**

Potentially useful formulas:

Let u and w denote positive real numbers, then:

(a) $\log_b(uv) = \log_b(u) + \log_b(v)$

(b) $\log_b\left(\frac{u}{v}\right) = \log_b(u) - \log_b(v)$

(c) $\log_b(u^c) = c\log_b(u)$ where c is any real number.

(d) $\log_b(u) = \dfrac{\log_a(u)}{\log_a(b)}$ for $a > 0$, $a \neq 1$.

(e) $A = \frac{1}{2}r^2\theta$

(f) $S = r\theta$

NOTE: YOU MAY TEAR OFF THIS FIRST PAGE AND USE (FRONT AND BACK) AS SCRATCH PAPER.

i. DO NOT START UNTIL INSTRUCTED BY A PROCTOR.

ii. THE EXAM IS ON BOTH SIDES OF EACH FOLLOWING EXAM PAGE

iii. WRITE YOUR NAME ON THE NEXT PAGE. JUST BEFORE YOU UPLOAD TO GRADESCOPE WRITE DOWN YOUR UPLOAD TIME ON THE NEXT PAGE.

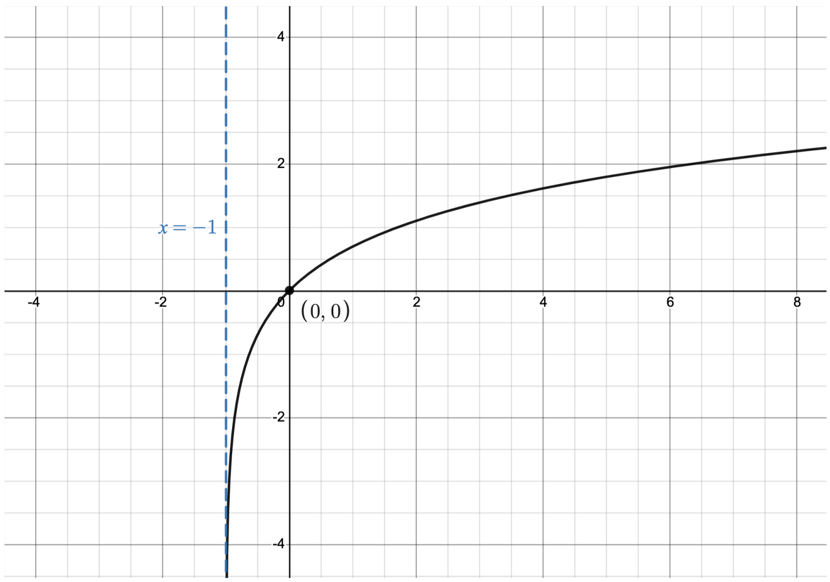
iv. WHEN YOU FINISH (IF BEFORE THE EXAM END TIME) PLEASE QUIETLY COLLECT YOUR THINGS AND LINE UP AT THE BACK OF THE ROOM. A PROCTOR WILL INDICATE WHEN IT'S YOUR TURN TO EXIT THE ROOM AND UPLOAD TO GRADESCOPE.

Name: ______________________________ Upload time: __________

1. Sketch the following graphs: Be sure to label any asymptotes and intercepts for each graph.

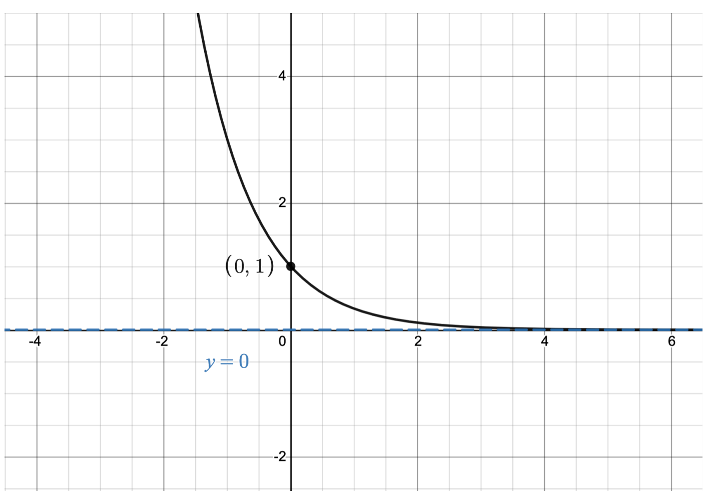
   (a) $f(x) = \ln(x + 1)$ (4 pts)

   Solution



   (b) $g(x) = 3^{-x}$ (4 pts)

   Solution



   (c) What is the domain of $f(x)$ in part (a) (2 pts)? (3 pts)

   Solution

   From the graph the domain is $(-1, \infty)$. Alternatively, note that we can find the domain by setting $x + 1 > 0$. Solving this inequality we get $x > -1$ resulting in domain $(-1, \infty)$.

2. (a) Simplify (rewrite without logs): $-\log_4(64) - e^{\ln(5)} + \log_5(25^x) + \log(1)$ (4 pts)

Solution

$-\log_4(64) - e^{\ln(5)} + \log_5(25^x) + \log(1) = -3 - 5 + x\log_5(25) + 0 = -8 + x(2) = 2x - 8.$

(b) Rewrite as a single logarithm without negative exponents: $2\log(x) - \frac{1}{4}\log(x) + 3\log(y)$ (4 pts)

Solution

$$
\begin{aligned}
2\log(x) - \frac{1}{4}\log(x) + 3\log(y) & \qquad (1)\\
= \log\left(x^2\right) - \log\left(x^{1/4}\right) + \log\left(y^3\right) & \qquad (2)\\
= \log\left(\frac{x^2}{x^{1/4}}\right) + \log\left(y^3\right) & \qquad (3)\\
= \log\left(\frac{x^2 y^3}{x^{1/4}}\right) & \qquad (4)\\
= \log\left(x^{7/4}y^3\right) & \qquad (5)
\end{aligned}
$$

(c) Rewrite as a sum/difference of logarithms without any exponents: $\ln\left(\frac{xe^x}{\sqrt[3]{yz}}\right)$ (4 pts)

Solution

$$
\begin{aligned}
\ln\left(\frac{xe^x}{\sqrt[3]{yz}}\right) & \qquad (6)\\
= \ln\left(xe^x\right) - \ln(\sqrt[3]{yz}) & \qquad (7)\\
= \ln(x) + \ln\left(e^x\right) - \ln\left((yz)^{1/3}\right) & \qquad (8)\\
= \ln(x) + x - \frac{1}{3}\ln(yz) & \qquad (9)\\
= \ln(x) + x - \frac{1}{3}\left(\ln(y) + \ln(z)\right) & \qquad (10)\\
= \ln(x) + x - \frac{1}{3}\ln(y) - \frac{1}{3}\ln(z) & \qquad (11)
\end{aligned}
$$

3. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit).

(a) $4^{5-x} = 16$ (4 pts)

Solution

$$
\begin{aligned}
4^{5-x} &= 16 & \qquad (12)\\
\log_4\left(4^{5-x}\right) &= \log_4(16) & \qquad (13)\\
5 - x &= 2 & \qquad (14)\\
x &= 3 & \qquad (15)
\end{aligned}
$$

(b) $\log_3(x) = 2$ (4 pts)

Solution

$$\log_3(x) = 2 \tag{16}$$
$$3^2 = x \tag{17}$$
$$x = 9 \tag{18}$$

Checking $x = 9$ in the original equation we see this is indeed a solution of the original equation.

(c) $3^{2x+1} = 2$ (4 pts)

Solution

$$3^{2x+1} = 2 \tag{19}$$
$$\log_3\left(3^{2x+1}\right) = \log_3(2) \tag{20}$$
$$2x + 1 = \log_3(2) \tag{21}$$
$$2x = \log_3(2) - 1 \tag{22}$$
$$x = \frac{\log_3(2) - 1}{2} \tag{23}$$

(d) $\ln(2) + \ln(x) = \ln\left(x^2 - 3\right)$ (4 pts)

Solution

$$\ln(2) + \ln(x) = \ln\left(x^2 - 3\right) \tag{24}$$
$$\ln(2x) = \ln\left(x^2 - 3\right) \tag{25}$$
$$2x = x^2 - 3 \tag{26}$$
$$0 = x^2 - 2x - 3 \tag{27}$$
$$0 = (x + 1)(x - 3) \tag{28}$$

This gives us potential solutions $x = -1, 3$. Checking both potential solutions in the original equation we see that only $x = 3$ is a solution.

(e) $\log(x+1) = 3\log\left(2x^{1/3}\right)$ (4 pts)

Solution

$$\begin{aligned}
\log(x+1) &= 3\log\left(2x^{1/3}\right) && (29)\\
\log(x+1) &= \log\left(\left(2x^{1/3}\right)^3\right) && (30)\\
\log(x+1) &= \log(8x) && (31)\\
x+1 &= 8x && (32)\\
1 &= 7x && (33)\\
x &= \frac{1}{7} && (34)
\end{aligned}$$

Checking $x = \frac{1}{7}$ in the original equation we see this is in fact a solution.

4. A hot cup of coffee is left in a room. It's temperature in Fahrenheit, $T(t)$, at time t in hours, is expected to cool according to the exponential model:

$$T(t) = 66 + 112e^{-0.05t}$$

Answer the following:

(a) Find the initial temperature of the cup of coffee. (3 pts)

Solution

The initial temperature is found when $t = 0$. $T(0) = 66 + 112e^{-0.05(0)} = 66 + 112 = 178^\circ$ F.

(b) Find the temperature of the cup of coffee after 5 hours. (3 pts)

Solution

The temperature at $t = 5$ is found by. $T(5) = 66 + 112e^{-0.05(5)} = 66 + 112e^{-1/4}\,{}^\circ$ F.

(c) According to the model is it possible for the cup of coffee to cool to 66° F? If so, find the time t which gives a temperature of 66° F. Be sure to justify your answer. (3 pts)

Solution

In order for the cup of coffee to cool to 66° F then there must be a time, t, such that $66 + 112e^{-0.05t} = 66$. Solving for t we get:

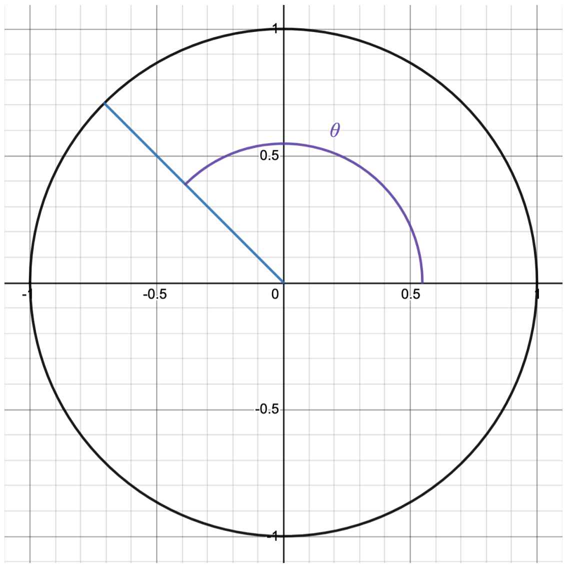
$$\begin{aligned}
66 + 112e^{-0.05t} &= 66 && (35)\\
112e^{-0.05t} &= 0 && (36)\\
e^{-0.05t} &= 0 && (37)
\end{aligned}$$

For which there is no t value such that $e^{-0.05t}$ equals zero. According to the model the cup of coffee will never cool to 66° F.

5. Sketch each angle in standard position on the unit circle.

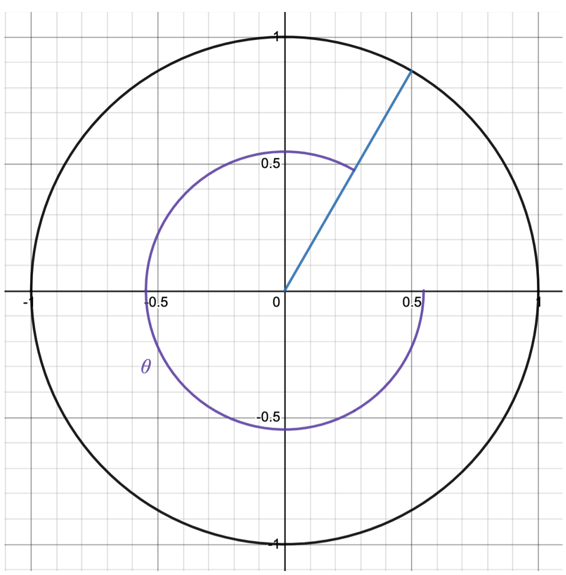
   (a) $\dfrac{3\pi}{4}$ (3 pts)

   Solution



   (b) $-\dfrac{5\pi}{3}$ (3 pts)

   Solution



6. Let $\left(x, \dfrac{1}{3}\right)$ be a point on the unit circle that lies on the terminal side of an angle θ in standard position. Suppose we also know $\cos\theta < 0$. Use this information to answer the following:

   (a) Considering all given information, what quadrant does θ to lie in? (4 pts)

   Solution

   θ lies in quadrant II since the y-coordinate is positive in quadrants I and II and $\cos\theta < 0$ in quadrants II and III.

(b) Find the value for x. (4 pts)

Solution

Since the point lies on the unit circle we know that the point satisfies the equation $x^2 + y^2 = 1$. Plugging in $y = \frac{1}{3}$ we can solve for x:

$$x^2 + \left(\frac{1}{3}\right)^2 = 1 \tag{38}$$

$$x^2 = 1 - \frac{1}{9} \tag{39}$$

$$x^2 = \frac{8}{9} \tag{40}$$

$$x = \pm\frac{2\sqrt{2}}{3} \tag{41}$$

Since the point lies in quadrant II then $x = -\frac{2\sqrt{2}}{3}$.

(c) Find $\sin\theta$ (4 pts)

Solution

On the unit circle: $\sin\theta = y = \frac{1}{3}$.

(d) Find $\csc\theta$ (2 pts)

Solution

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{1}{3}} = 3.$

7. Find the exact value of each of the following. If a value does not exist write DNE.

   (a) $\cos\left(\frac{3\pi}{4}\right)$ (4 pts)

   Solution

   $$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

   (b) $\sin(90°)$ (4 pts)

   Solution

   $$\sin(90°) = 1.$$

   (c) $\tan\left(-\frac{5\pi}{4}\right)$ (4 pts)

   Solution

   $$\tan\left(-\frac{5\pi}{4}\right) = -1$$

   (d) $\sec(60°)$ (4 pts)

   Solution

   $$\sec(60°) = \frac{1}{\cos(60°)} = \frac{1}{\frac{1}{2}} = 2$$

   (e) $\csc\left(\frac{7\pi}{6}\right)$ (4 pts)

   Solution

   $$\csc\left(\frac{7\pi}{6}\right) = \frac{1}{\sin\left(\frac{7\pi}{6}\right)} = \frac{1}{-\frac{1}{2}} = -2$$

8. The radius of the sector of a circle with a central angle of $70°$ is 2 inches. Find the area of the sector. (4 pts)

   Solution

   We can use $A = \frac{1}{2}r^2\theta$ but this formula only works if θ is in radians. $\theta = (70°)\left(\frac{\pi}{180°}\right) = \frac{7\pi}{18}$. So $A = \frac{1}{2}(2)^2\left(\frac{7\pi}{18}\right) = \frac{7\pi}{9}$ in^2.

9. You are standing in a flat non-sloping backyard. You measure the angle from your feet to the top of a house to be $30°$. You know your distance to the base of the house is 50 ft. How high is the top of the house? (5 pts)

   Solution

   Let h be the height of the house. We can write: $\tan(30°) = \frac{h}{50}$. Since $\tan(30°) = \frac{\sin(30°)}{\cos(30°)} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$ then solving for h we get: $h = 50\tan(30°) = \frac{50}{\sqrt{3}}$ ft.