INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **justify all answers**. A correct answer with incorrect work or no justification may receive no credit. Books, notes, and electronic devices are not permitted while taking the exam. The exam is worth 100 points.

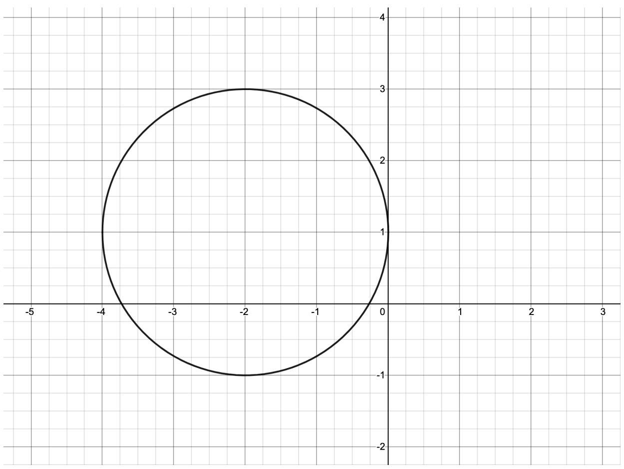
Potentially useful formulas:

(i) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

(ii) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

(iii) Equation of a circle: $(x-h)^2 + (y-k)^2 = r^2$

1. For the graph of the circle, answer the following: (7 pts)



(a) Identify the radius and the coordinates of the center of the circle.

Solution:

From the graph we see the circle is centered at $(-2, 1)$. The radius is $r = 2$.

(b) Write down the equation of the circle.

Solution:

The general form of a circle is $(x-h)^2 + (y-k)^2 = r^2$ with center (h, k) and radius r. For this circle the equation is: $(x - (-2))^2 + (y-1)^2 = 2^2$ simplifying to $(x+2)^2 + (y-1)^2 = 4$.

2. Find the domain of the following functions. Express your answers in interval notation. (15 pts)

(a) $s(x) = \dfrac{x^2 - 4x + 3}{x^2 - 3x}$

Solution:

The domain is all real numbers except when $x^2 - 3x = 0$. We solve the equation:

$$\begin{align} x^2 - 3x &= 0 \tag{1} \\ x(x-3) &= 0 \tag{2} \end{align}$$

This results in a domain of all real numbers but we must exclude $x = 0$ and $x = 3$. In interval notation: $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$.

(b) $h(x) = \dfrac{\sqrt{x-3}}{2x - 8}$

Solution:

The domain is found when $x - 3 \geq 0$ resulting in $x \geq 3$. We must also exclude the x-value that results in $2x - 8 = 0$ which is $x = 4$. Thus the domain is $[3, 4) \cup (4, \infty)$.

(c) $n(t) = 4t + \sqrt[3]{t-1}$

Solution:

$n(t)$ has a domain of all real numbers since there is no possibility of division by zero or taking the even root of a negative number. The domain is $(-\infty, \infty)$.

3. For $f(x) = 2x^2 - x$ answer the following: (10 pts)

(a) Find $f(a)$

Solution:

$f(a) = 2a^2 - a$

(b) Find $f(a+h)$

Solution:

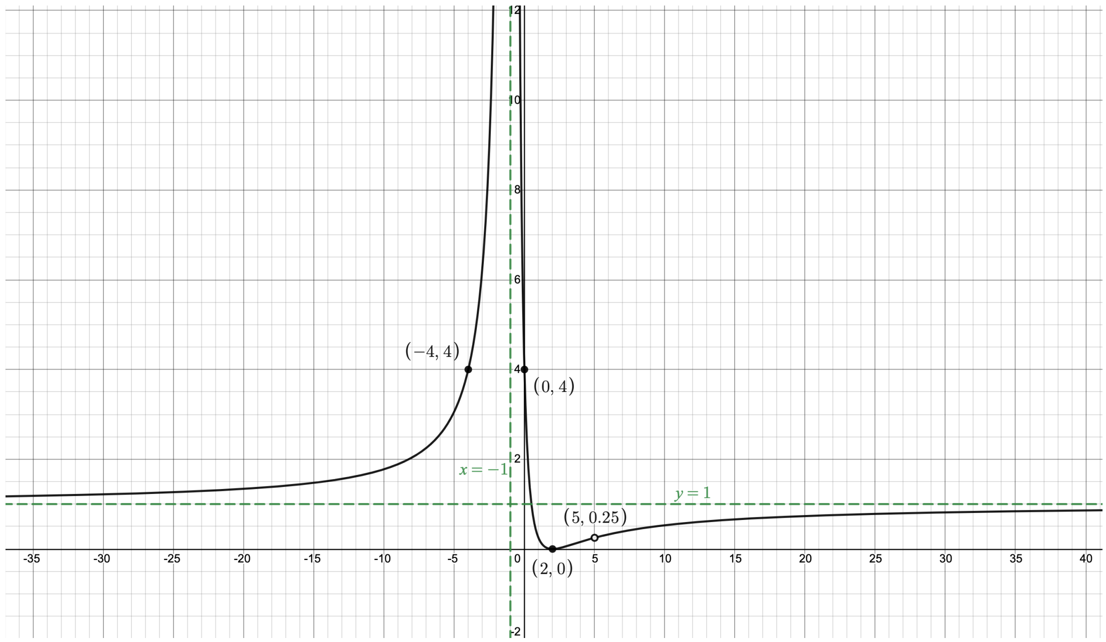
$$\begin{aligned} f(a+h) &= 2(a+h)^2 - (a+h) && (3) \\ &= 2(a^2 + 2ah + h^2) - a - h && (4) \\ &= 2a^2 + 4ah + 2h^2 - a - h && (5) \end{aligned}$$

(c) Find $\dfrac{f(a+h) - f(a)}{h}$ and simplify.

Solution:

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{2a^2 + 4ah + 2h^2 - a - h - (2a^2 - a)}{h} && (6) \\ &= \frac{2a^2 + 4ah + 2h^2 - a - h - 2a^2 + a}{h} && (7) \\ &= \frac{4ah + 2h^2 - h}{h} && (8) \\ &= \frac{h(4a + 2h - 1)}{h} && (9) \\ &= 4a + 2h - 1 && (10) \end{aligned}$$

4. Answer the following for the given graph of a rational function $R(x)$ with labeled asymptotes. (15 pts)



(a) Find the domain of $R(x)$. Express your answer in interval notation.

Solution:

From the graph: $(-\infty, -1) \cup (-1, 5) \cup (5, \infty)$

(b) Find the range of $R(x)$. Express your answer in interval notation.

Solution:

From the graph: $[0, \infty)$

(c) Find the x-intercept(s) of $R(x)$.

Solution:

From the graph: $(2, 0)$

(d) Find $(R \cdot R)(-4)$.

Solution:

$(R \cdot R)(-4) = R(-4) \cdot R(-4) = 4 \cdot 4 = 16.$

(e) Find $(R \circ R)(2)$.

Solution:

$(R \circ R)(2) = R(R(2)) = R(0) = 4.$

(f) $R(x)$ is **not** one-to-one. Give a brief explanation as to why $R(x)$ is not one-to-one.

Solution:

$R(x)$ is not one-to-one since $R(0) = 4 = R(-4)$ but $0 \neq -4$. This is referred to as not passing the horizontal line test.

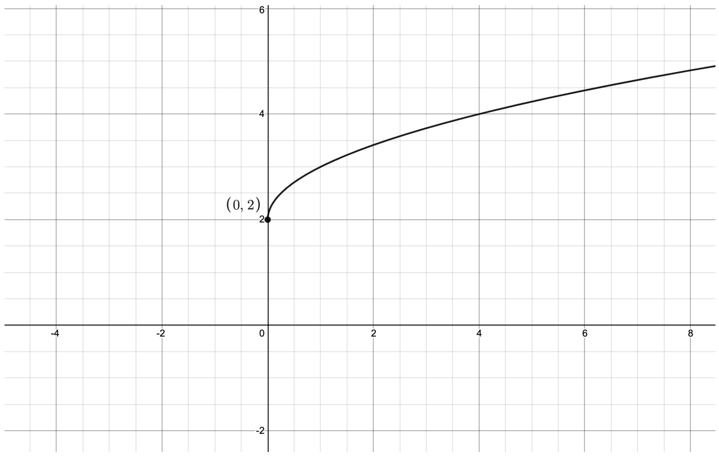
(g) Write down a restriction of the domain of $R(x)$ so that the function on the restricted domain has the same range and is one-to-one.

Solution:

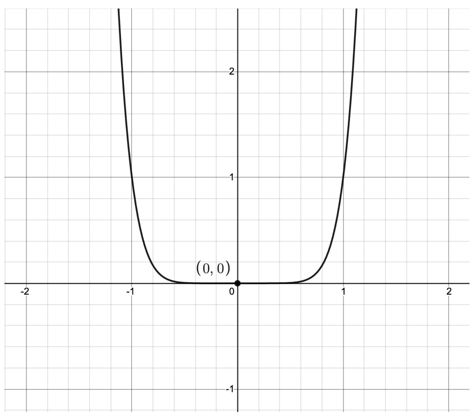
Restrict the domain to $(-1, 2]$. The function on the restricted domain has range $[0, \infty)$ and passes the horizontal line test.

5. Sketch the shape of the graph of each of the following on the given set of axes. Label x or y-intercepts if any: (19 pts)

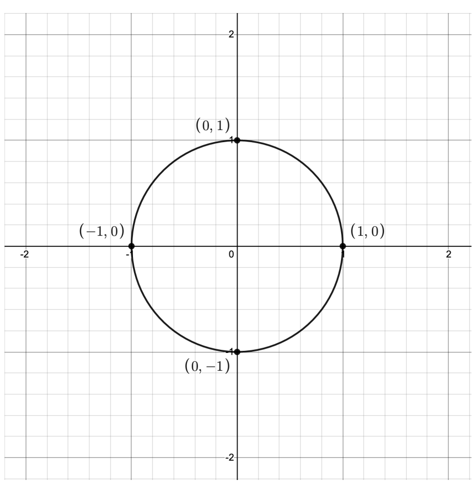
(a) $f(x) = \sqrt{x} + 2$



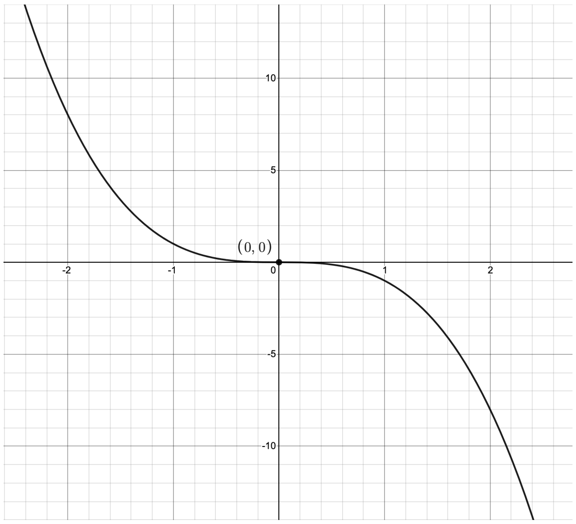
(b) $k(x) = x^8$



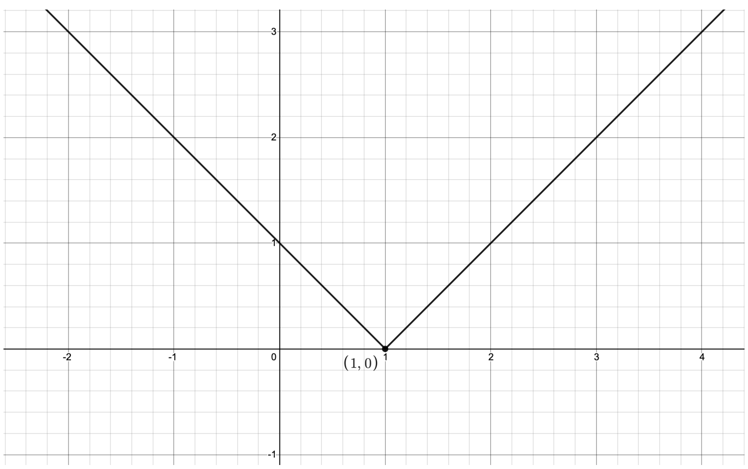
(c) $x^2 + y^2 = 1$



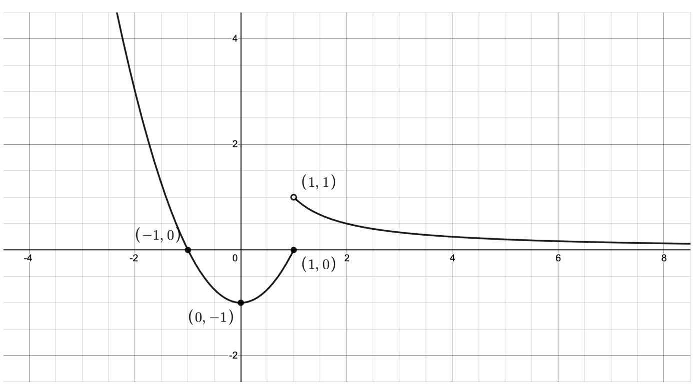
(d) $g(x) = -x^3$



(e) $m(x) = |x - 1|$



(f) $q(x) = \begin{cases} x^2 - 1 & \text{if} \quad x \leq 1 \\ \frac{1}{x} & \text{if} \quad x > 1 \end{cases}$



6. Find the inverse function of $g(x) = 1 - 3x^3$ (you may assume the inverse function exists). (5 pts)

Solution:

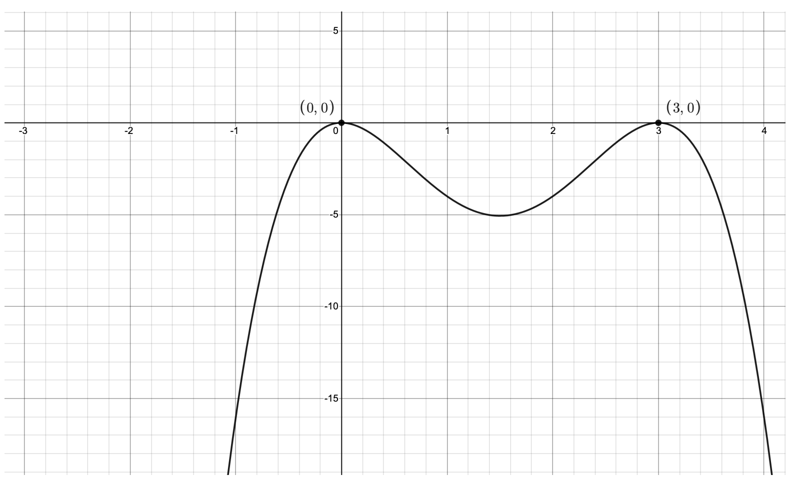
$$y = 1 - 3x^3 \tag{11}$$
$$y - 1 = -3x^3 \tag{12}$$
$$\frac{y-1}{-3} = x^3 \tag{13}$$
$$\sqrt[3]{\frac{y-1}{-3}} = x \tag{14}$$

Swapping x and y and replacing y by $g^{-1}(x)$ we get $g^{-1}(x) = \sqrt[3]{-\frac{x-1}{3}}$ or $g^{-1}(x) = -\sqrt[3]{\frac{x-1}{3}}$.

7. (a) Sketch the shape of the graph of a polynomial function, $P(x)$, that satisfies **all** of the information. **Label** all intercepts on the graph. (6 pts)

   i. The graph has y-intercept $(0, 0)$.
   ii. The graph has end behavior consistent with $y = -x^4$
   iii. The graph bounces (touches but does not cross) at x-intercept $(3, 0)$
   iv. The graph has no other x-intercepts.

Solution:



(b) Write down a polynomial $P(x)$ that satisfies all of the given information. (4 pts)

Solution:

$P(x) = -x^2(x-3)^2$

8. Use long division to find the quotient and remainder when $x^3 - x^2 + 2x + 1$ is divided by $x^2 + 2x$. (6 pts)

Solution:

$$
\begin{array}{r}
x - 3 \\
x^2 + 2x \, \overline{\smash{\big)}\, x^3 - x^2 + 2x + 1} \\
-(\underline{x^3 + 2x^2)} \\
-3x^2 + 2x + 1 \\
-(\underline{-3x^2 - 6x)} \\
8x + 1
\end{array}
$$

So the quotient is $x - 3$ and the remainder is $8x + 1$.

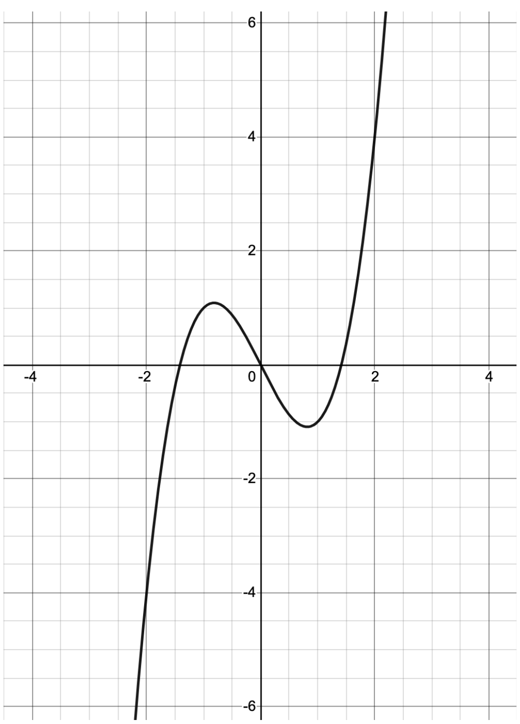
9. (a) Is $f(x) = x^5 - \sqrt[3]{x}$ odd, even, or neither? Justify your answer for credit. (4 pts)

Solution:

$$
\begin{aligned}
f(-x) &= (-x)^5 - \sqrt[3]{-x} && (15)\\
&= -x^5 + \sqrt[3]{x} && (16)\\
&= -(x^5 - \sqrt[3]{x}) && (17)\\
&= -f(x) && (18)
\end{aligned}
$$

So the function is odd.

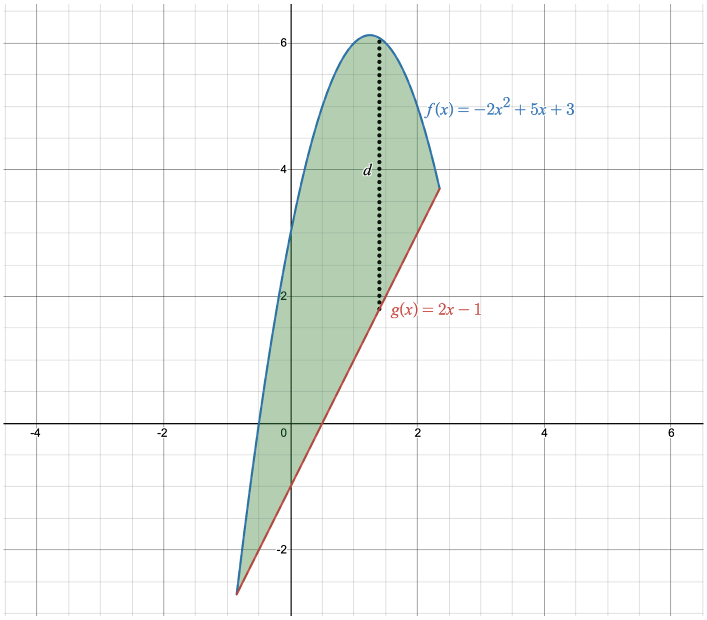
(b) Given the graph of a function below, is the function odd, even, or neither? **No justification is needed.** (4 pts)



Solution:

This is the graph of a function symmetric about the origin and is therefore odd.

10. Find the maximum vertical distance d between the parabola, $f(x) = -2x^2 + 5x + 3$, and the line, $g(x) = 2x - 1$, for the green region. (5 pts)



The vertical distance is found by subtracting $g(x)$ from $f(x)$. The distance, $D(x)$, is: $D(x) = f(x) - g(x) = -2x^2 + 5x + 3 - (2x - 1) = -2x^2 + 3x + 4$. The maximum of the distance is found by finding the vertex of $D(x) = -2x^2 + 3x + 4$ since the graph of $D(x)$ is a downward pointing parabola. The x-coordinate is found by using the vertex formula: $h = -\dfrac{b}{2a} = -\dfrac{3}{2(-2)} = \dfrac{3}{4}$. So the maximum is found at $D\left(\dfrac{3}{4}\right) = -2\left(\dfrac{3}{4}\right)^2 + 3\left(\dfrac{3}{4}\right) + 4 = -\dfrac{9}{8} + \dfrac{9}{4} + 4 = -\dfrac{9}{8} + \dfrac{18}{8} + \dfrac{32}{8} = \dfrac{41}{8}$.

Note: The distance formula can be used here: If $(x, f(x))$ is a point on the graph of $f(x)$ and $(x, g(x))$ is a point on the graph of $g(x)$ (they share the same x-coordinate since the points are directly above each other) then $D(x) = \sqrt{(f(x) - g(x))^2 + (x - x)^2} = |f(x) - g(x)| = f(x) - g(x)$ since $f(x) \geq g(x)$ for all x-values in the shaded region.