1. Add/subtract/divide/multiply as indicated: (6 pts)

(a) $\frac{7}{3} - \frac{\frac{3}{2}}{9} + 3^{-1}$

Solution:

$$\begin{aligned}
\frac{7}{3} - \frac{\frac{3}{2}}{9} + 3^{-1} &= \frac{7}{3} - \frac{3}{2} \cdot \frac{1}{9} + \frac{1}{3} && (1)\\
&= \frac{8}{3} - \frac{1}{6} && (2)\\
&= \frac{16}{6} - \frac{1}{6} && (3)\\
&= \frac{15}{6} && (4)\\
&= \boxed{\frac{5}{2}} && (5)
\end{aligned}$$

(b) $(e^x)^3 - e^x\left(e^{2x} - 2e^{-x}\right)$

Solution:

$$\begin{aligned}
(e^x)^3 - e^x\left(e^{2x} - 2e^{-x}\right) &= e^{3x} - e^x \cdot e^{2x} + e^x \cdot 2e^{-x} && (6)\\
&= e^{3x} - e^{3x} + 2e^0 && (7)\\
&= e^{3x} - e^{3x} + 2 && (8)\\
&= \boxed{2} && (9)
\end{aligned}$$

2. The following are unrelated. (10 pts)

(a) Place the correct symbol, $<$, $>$, or, $=$ in the space between each of the following pair of numbers.

i. $\frac{5}{12} \quad \frac{4}{9}$

Solution:

The numbers are more easily compared if we get them to have the same denominator:

$\frac{4}{9} = \frac{16}{36}$ and $\frac{5}{12} = \frac{15}{36}$

We see then that: $\frac{15}{36} < \frac{16}{36}$ and so $\frac{5}{12}\boxed{<}\frac{4}{9}$

ii. $-\sqrt{2} \quad -\sqrt{3}$

Solution:

Since $\sqrt{2} < \sqrt{3}$ then we can multiply both sides by -1 and, flipping the inequality, we get $-\sqrt{2}\boxed{>}-\sqrt{3}$

(b) Let x and y be real numbers such that $x < 0$ and $y < 0$. Determine whether the following expression is positive, negative, or zero: $-\dfrac{x^7}{8y^3}$.

Solution:

x^7 is negative when x is negative.

y^3 is negative when y is negative.

So $\dfrac{x^7}{8y^3}$ is positive and thus $-\dfrac{x^7}{8y^3}$ is $\boxed{\text{negative}}$.

(c) Rewrite the following without absolute value symbols:

i. $|e - 1|$

Solution:

Since $e - 1$ is positive then we can simply drop the inequality: $|e - 1| = \boxed{e - 1}$.

ii. $|x - 3|$ where $x < 3$

Solution:

Since x is less than 3 then $x - 3$ is negative. In order to remove the absolute value, we need to make $x - 3$ positive so we multiply $x - 3$ by -1:

$|x - 3| = \boxed{-(x - 3)}$

3. The following are unrelated. Leave answers without negative exponents. (6 pts)

(a) Simplify: $(2r - 3)^2 - \dfrac{54\left(3r^2\right)^{-3}}{r^{-7}}$

Solution:

$$\begin{aligned}
(2r - 3)^2 - \frac{54\left(3r^2\right)^{-3}}{r^{-7}} &= 4r^2 - 12r + 9 - 54(3^{-3} \cdot r^{-6})r^7 && (10)\\
&= 4r^2 - 12r + 9 - \frac{54}{27} \cdot r && (11)\\
&= \boxed{4r^2 - 14r + 9} && (12)
\end{aligned}$$

(b) Evaluate and simplify: $\dfrac{\cos(\frac{\pi}{3})}{\sqrt{2}} - 8^{-1/2}$

Solution:

$$\begin{aligned}
\frac{\cos(\frac{\pi}{3})}{\sqrt{2}} - 8^{-1/2} &= \frac{\frac{1}{2}}{\sqrt{2}} - \frac{1}{\sqrt{8}} && (13)\\
&= \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} && (14)\\
&= \boxed{0} && (15)
\end{aligned}$$

4. The following are unrelated. (6 pts)

(a) Simplify: $\dfrac{x^2+1}{x^2-1} \cdot \dfrac{x^2-3x+2}{3x^4+3x^2}$

Solution:

$$\begin{aligned} \frac{x^2+1}{x^2-1} \cdot \frac{x^2-3x+2}{3x^4+3x^2} &= \frac{x^2+1}{(x+1)(x-1)} \cdot \frac{(x-1)(x-2)}{3x^2(x^2+1)} && (16) \\ &= \boxed{\frac{x-2}{3x^2(x+1)}} && (17) \end{aligned}$$

(b) Multiply and simplify: $\left(\sqrt{x}-3\sqrt{y}\right)\left(\sqrt{x}+3\sqrt{y}\right)$

Solution:

$$\begin{aligned} \left(\sqrt{x}-3\sqrt{y}\right)\left(\sqrt{x}+3\sqrt{y}\right) &= \sqrt{x}\sqrt{x} - \sqrt{x}3\sqrt{y} - 3\sqrt{y}\sqrt{x} + 3\sqrt{y}3\sqrt{y} && (18) \\ &= \boxed{x-9y} && (19) \end{aligned}$$

5. The following are unrelated. (6 pts)

(a) Simplify: $\dfrac{1+\frac{1}{x-2}}{\frac{x}{x-2}-\frac{1}{x-2}}$

Solution:

$$\begin{aligned} \frac{1+\frac{1}{x-2}}{\frac{x}{x-2}-\frac{1}{x-2}} &= \frac{\frac{x-2}{x-2}+\frac{1}{x-2}}{\frac{x}{x-2}-\frac{1}{x-2}} && (20) \\ &= \frac{\frac{x-1}{x-2}}{\frac{x-1}{x-2}} && (21) \\ &= \boxed{1} && (22) \end{aligned}$$

(b) Simplify the following: $\log(1) + \dfrac{\log_2\left(\frac{1}{2}\right)}{\log_2(8)} - e^{\ln 4}$ (Your answer should have no logarithms)

Solution:

$$\begin{aligned} \log(1) + \frac{\log_2\left(\frac{1}{2}\right)}{\log_2(8)} - e^{\ln 4} &= 0 + \frac{-\log_2(2)}{\log_2(2^3)} - 4 && (23) \\ &= \frac{-\log_2(2)}{3\log_2(2)} - 4 && (24) \\ &= -\frac{1}{3} - 4 && (25) \\ &= \boxed{-\frac{13}{3}} && (26) \end{aligned}$$

6. The following are unrelated: (7 pts)

(a) Multiply and simplify: $(2-3i)(-1+i)$

Solution:

$$\begin{aligned}
(2-3i)(-1+i) &= -2+2i+3i-3i^2 && (27)\\
&= -2+5i-3(-1) && (28)\\
&= -2+5i+3 && (29)\\
&= \boxed{1+5i} && (30)
\end{aligned}$$

(b) Solve the following equation over the complex numbers. $x^2+4x+5=0$

Solution:

$$\begin{aligned}
x &= \frac{-4 \pm \sqrt{(4)^2-4(1)(5)}}{2(1)} && (31)\\
&= \frac{-4 \pm \sqrt{16-20}}{2} && (32)\\
&= \frac{-4 \pm \sqrt{-4}}{2} && (33)\\
&= \frac{-4 \pm 2i}{2} && (34)\\
&= \boxed{-2 \pm i} && (35)
\end{aligned}$$

7. Solve the following equations for the indicated variable. If there are no solutions, write **no solutions**. (12 pts)

(a) Solve for a: $\frac{5}{6}a - \frac{1}{2} = \frac{a}{2} - \frac{2}{3}$

Solution: We can first get rid of the fractions by multiplying both sides of the equality by the least common denominator, which in this case is 6.

$$\begin{aligned}
6 \cdot \left(\frac{5}{6}a - \frac{1}{2}\right) = \left(\frac{a}{2} - \frac{2}{3}\right) \cdot 6 && (36)\\
5a - 3 = 3a - 4 && (37)\\
2a = -1 && (38)\\
\boxed{a = -\frac{1}{2}} && (39)
\end{aligned}$$

(b) Solve for x: $\sqrt{x^2+4}+2=2x$

Solution: We first want to get the radical alone on one side of the equality, then square both sides to get rid of the square root.

$$\sqrt{x^2+4}+2=2x \tag{40}$$
$$\sqrt{x^2+4}=2x-2 \tag{41}$$
$$(\sqrt{x^2+4})^2=(2x-2)^2 \tag{42}$$
$$x^2+4=4x^2-8x+4 \tag{43}$$
$$0=3x^2-8x \tag{44}$$
$$0=x(3x-8) \tag{45}$$
$$x=0 \text{ and } x=\frac{8}{3} \tag{46}$$

Since we started with a radical equation, we must check our solutions.
For $x=0$:

$$\sqrt{0^2+4}+2\neq 2(0) \;\; \text{X}$$

And for $x=\frac{8}{3}$, note that the Right Hand Side of the original equation simplifies to $2\cdot\frac{8}{3}=\frac{16}{3}$ so we check the Left Hand Side:

$$\sqrt{\left(\frac{8}{3}\right)^2+4}+2=\sqrt{\frac{64}{9}+4}+2 \tag{47}$$
$$=\sqrt{\frac{64}{9}+\frac{36}{9}}+2 \tag{48}$$
$$=\sqrt{\frac{100}{9}}+2 \tag{49}$$
$$=\frac{10}{3}+2 \tag{50}$$
$$=\frac{10}{3}+\frac{6}{3} \tag{51}$$
$$=\frac{16}{3}\checkmark \tag{52}$$

Therefore, the only solution to this equation is $\boxed{x=\frac{8}{3}}$.

(c) Solve for t: $\ln(t)=\ln(2)+\ln(t+2)$

Solution:

$$\ln(t)=\ln(2)+\ln(t+2) \tag{53}$$
$$\ln(t)=\ln(2(t+2)) \tag{54}$$
$$\ln(t)=\ln(2t+4) \tag{55}$$
$$t=2t+4 \tag{56}$$
$$t=-4 \tag{57}$$

Since we started with a logarithmic equation, we must check our solutions. Plugging in $t=-4$, would give us $\ln(-4)$ which is undefined as -4 is not in the domain of logarithmic functions. Therefore, there is $\boxed{\text{no solution.}}$

8. For the two points $P(5,-1)$ and $Q(2,3)$: (6 pts)

(a) Find the slope of the line through the two points.

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-1)}{5-2} = \boxed{-\frac{4}{3}}$$

(b) Find the equation of the line that passes through the points P and Q.

Solution:

Using the formula, $y = mx + b$, to find b we can use use either point given, using $Q(2,3)$ as the point

$$3 = -\frac{4}{3}(2) + b \tag{58}$$
$$3 = -\frac{8}{3} + b \tag{59}$$
$$b = \frac{17}{3} \tag{60}$$

Therefore, the equation of the line is $\boxed{y = -\frac{4}{3}x + \frac{17}{3}}$.

Alternate Solution:

Alternatively, we can use $y - y_1 = m(x - x_1)$:

$$y - 3 = -\frac{4}{3}(x-2) \tag{61}$$
$$y - 3 = -\frac{4}{3}x + \frac{8}{3} \tag{62}$$
$$\boxed{y = -\frac{4}{3}x + \frac{17}{3}} \tag{63}$$

9. Consider the functions: $f(x) = \dfrac{1}{x-1}$ and $g(x) = e^x$. (8 pts)

(a) Find $\left(\dfrac{g}{f}\right)(x)$.

Solution:

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} \tag{64}$$
$$= \frac{e^x}{\frac{1}{x-1}} \tag{65}$$
$$= \boxed{e^x(x-1)} \tag{66}$$

(b) Find $(f \circ g)(x)$.

Solution:

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) && (67)\\
&= f\left(e^{x}\right) && (68)\\
&= \boxed{\frac{1}{e^{x}-1}} && (69)
\end{aligned}
$$

(c) Find the domain of $(f \circ g)(x)$. Give your answer in interval notation.

Solution:

We start finding the domain by setting the denominator equal to zero:

$$
\begin{aligned}
e^{x}-1 &= 0 && (70)\\
e^{x} &= 1 && (71)\\
x &= \ln(1) && (72)\\
x &= 0 && (73)
\end{aligned}
$$

Since the domain of $g(x) = e^{x}$ is $(-\infty, \infty)$ then the domain of the composition is $\boxed{(-\infty, 0) \bigcup (0, \infty)}$

10. For the polynomial function $P(x) = 2(x-1)^{2}(x+2)$ (10 pts)

(a) Indicate on a graph or use arrow notation to indicate the end behavior of $P(x)$.

Solution: $\boxed{P(x) \to -\infty \text{ as } x \to -\infty \text{ and } P(x) \to \infty \text{ as } x \to \infty}$

(b) Find all zeros of $P(x)$ and identify the multiplicity for each.

Solution:

$$
\begin{aligned}
2(x-1)^{2}(x+2) &= 0 && (74)\\
(x-1)^{2}(x+2) &= 0 && (75)\\
(x-1)^{2} &= 0 \text{ or } x+2=0 && (76)\\
x &= 1 \text{ and } x=-2 && (77)
\end{aligned}
$$

So the zeros are $\boxed{x = 1 \text{ with multiplicity } 2 \text{ and } x = -2 \text{ with multiplicity } 1}$.

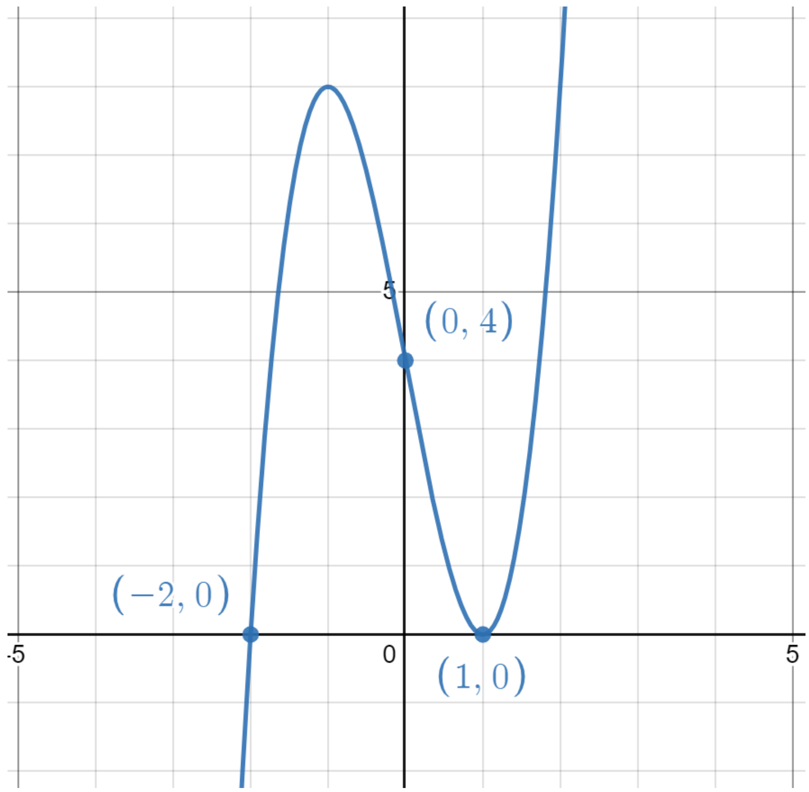
(c) Find the y-intercept.

Solution:

$$
\begin{aligned}
P(0) &= 2(0-1)^{2}(0+2) && (78)\\
&= 2(-1)^{2}(2) && (79)\\
&= 2(1)(2) && (80)\\
&= 4 && (81)
\end{aligned}
$$

So the y-intercept is $\boxed{(0, 4)}$

(d) Sketch the graph of $P(x)$ using parts (a) through (c).



11. For the rational function $R(x) = \dfrac{x^3 + 7x^2 + 12x}{x^2 - 3x}$ answer the following (10 pts):

(a) Find the x-coordinate of any hole(s). If there are no hole(s) write NONE.

Solution:

We start by factoring the numerator and denominator to see if any factors cancel.

$$\frac{(x+3)(x+4)x}{(x-3)x} = \frac{(x+3)(x+4)}{x-3} \tag{82}$$

Since x cancels between the numerator and denominator then there is a hole with x-coordinate of $\boxed{x = 0}$.

(b) Find the y-coordinate of any hole(s). If there are no hole(s) write NONE.

Solution:

Plugging in our value from part (a) into the simplified function we get the y-coordinate of the hole:

$$\frac{(0+3)(0+4)}{(0-3)} = \frac{(3)(4)}{(-3)} \tag{83}$$

$$= -4 \tag{84}$$

So the y-coordinate of the hole is $\boxed{-4}$ and the hole is located at $(0, -4)$.

(c) Determine the end behavior of $R(x)$.

Solution:

Since the degree in the numerator is exactly one higher than the degree in the denominator then there is a slant asymptote. To find the slant asymptote, we long divide:

$$\begin{array}{r|l} & x + 10 \\ \hline x-3 & x^2 + 7x + 12 \\ & -(x^2 - 3x) \\ \hline & \quad 10x + 12 \\ & \quad -(10x - 30) \\ \hline & \qquad 42 \end{array}$$

So the slant asymptote is given by the quotient $\boxed{y = x + 10}$.

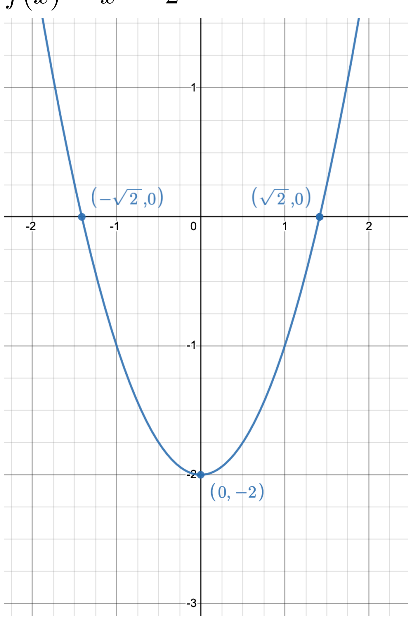
(d) Find all vertical asymptote(s). If there are none write NONE.

Solution:

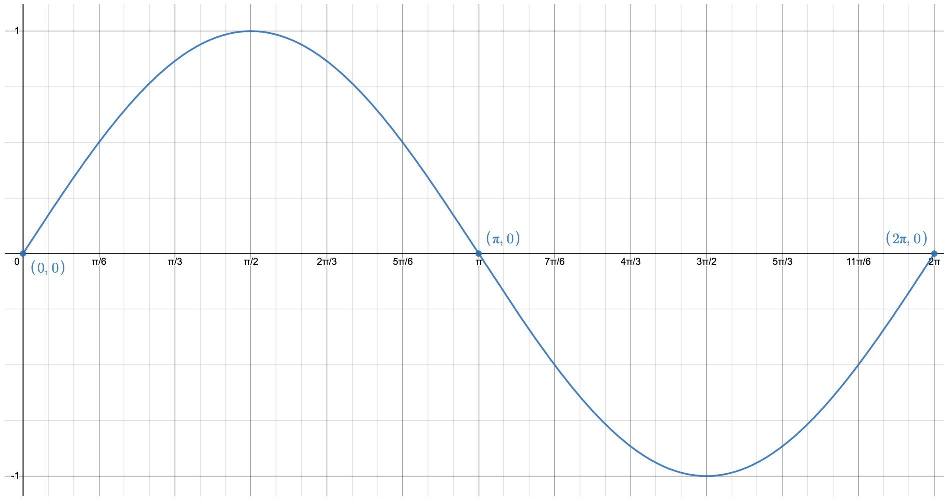
Now that we have the simplified rational function $y = \dfrac{x^2 + 7x + 12}{x - 3}$ the vertical asymptote is found by setting the denominator equal to zero. We get $x - 3 = 0$ so the vertical asymptote is $\boxed{x = 3}$.

12. Sketch the graph of the following functions. **Label all intercepts and asymptotes as appropriate.** (12 pts)

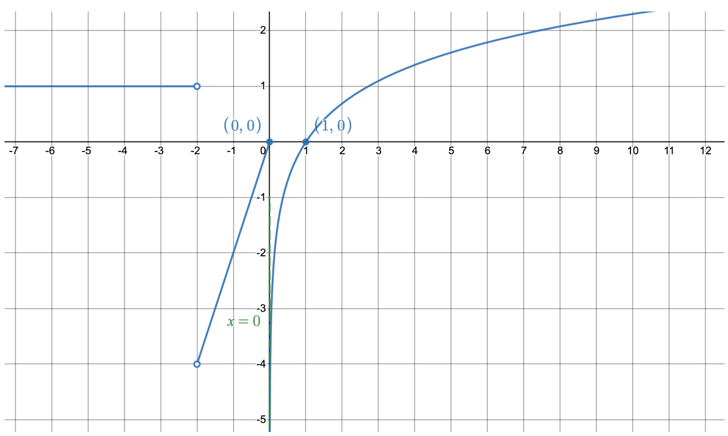
(a) $f(x) = x^2 - 2$



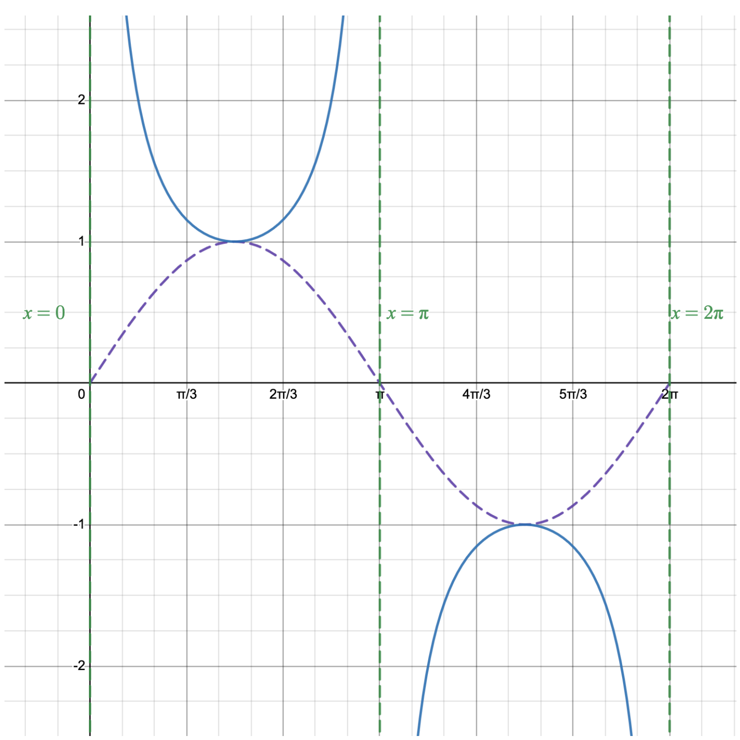
(b) $h(x) = \sin(x)$ on the restricted domain $[0, 2\pi]$



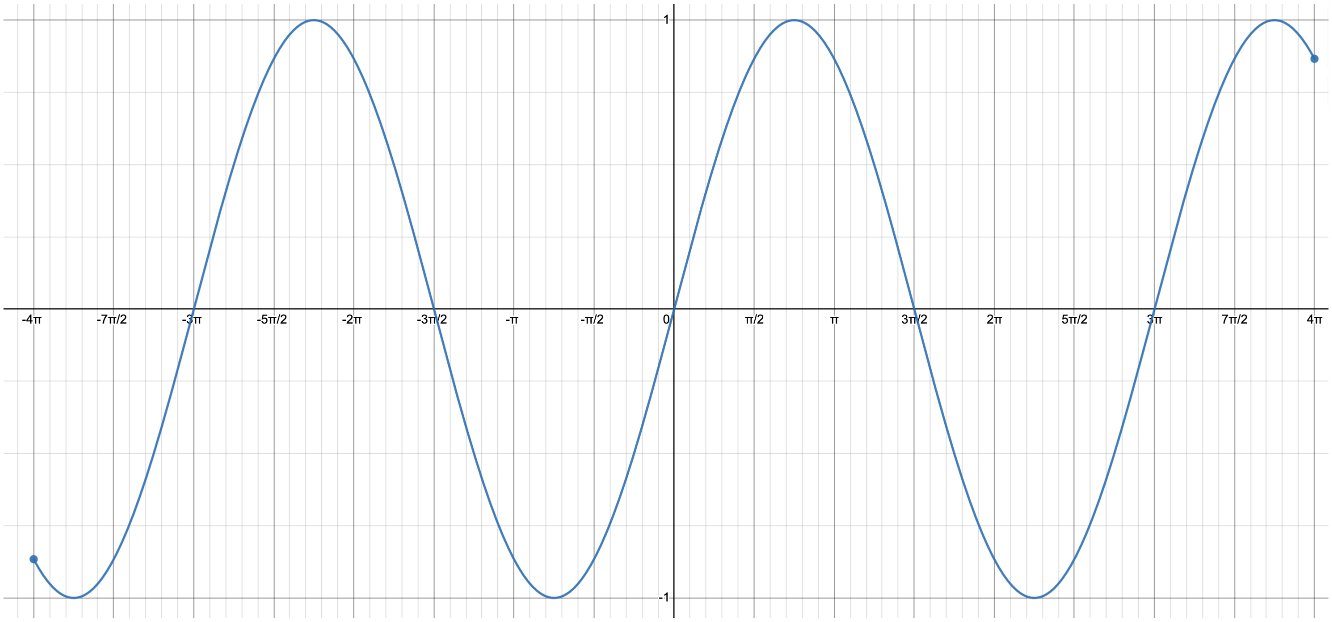
(c) $q(x) = \begin{cases} 1 & \text{if} \quad x \leq -2 \\ 2x & \text{if} \quad -2 < x \leq 0 \\ \ln(x) & \text{if} \quad x > 0 \end{cases}$



(d) $k(x) = \csc(x)$ on the restricted domain $[0, 2\pi]$



13. Use the graph of the function $T(x)$, with domain $[-4\pi, 4\pi]$, below to answer the following: (8 pts)



(a) Solve the equation $T(x) = 0$.

Solution: $\boxed{x = -3\pi, \dfrac{-3\pi}{2}, 0, \dfrac{3\pi}{2}, 3\pi}$

(b) Solve the inequality $T(x) > 0$. Give your answer in interval notation.

Solution: $\boxed{\left(-3\pi, \dfrac{-3\pi}{2}\right) \cup \left(0, \dfrac{3\pi}{2}\right) \cup (3\pi, 4\pi]}$

(c) Identify a restriction of the domain of $T(x)$ such that the range is preserved and the graph is one-to-one.

Solution: Any one of the following interval restrictions on the domain preserve the range:

$$\left[\frac{-15\pi}{4}, \frac{-9\pi}{4}\right] \text{ or } \left[\frac{-9\pi}{4}, \frac{-3\pi}{4}\right] \text{ or } \left[\frac{-3\pi}{4}, \frac{3\pi}{4}\right] \text{ or } \left[\frac{3\pi}{4}, \frac{9\pi}{4}\right] \text{ or } \left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$$

(d) Is $T(x)$ odd, even, or neither?

Solution:

Since $T(x)$ is symmetric about the origin, then the function is $\boxed{\text{odd}}$.

14. Find the exact value: (15 pts)

(a) $\sin(-\pi)$

Solution:

$\boxed{0}$

(b) $\cos\left(\frac{5\pi}{6}\right)$

Solution:

$\boxed{-\frac{\sqrt{3}}{2}}$

(c) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

Solution:

$\boxed{\frac{\pi}{4}}$

(d) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$

Solution:

$$\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \tag{85}$$
$$= \boxed{\frac{\pi}{6}} \tag{86}$$

(e) $\sin(75^\circ)$

Solution:

$$\sin(75^\circ) = \sin(30^\circ + 45^\circ) \tag{87}$$
$$= \sin(30^\circ)\cos(45^\circ) + \sin(45^\circ)\cos(30^\circ) \tag{88}$$
$$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} \tag{89}$$
$$= \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}} \tag{90}$$

(f) $\cos\left(\frac{\pi}{8}\right)$

Solution:

First note that $\frac{\pi}{8}$ is in quadrant I so $\cos\left(\frac{\pi}{8}\right)$ is positive. Now we apply the half angle formula:

$$\cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{\frac{\pi}{4}}{2}\right) \tag{91}$$
$$= \sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} \tag{92}$$
$$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \tag{93}$$
$$= \sqrt{\frac{2+\sqrt{2}}{4}} \tag{94}$$
$$= \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}} \tag{95}$$

15. Verify the identity: $\frac{\csc\theta - \sin\theta}{\sin\theta} = \cot^2\theta$. (4 pts)

Solution:

Starting with the Left Hand Side, we rewrite $\csc\theta$ in terms of $\sin\theta$ and then simplify the complex fraction by:

$$LHS: \frac{\csc\theta - \sin\theta}{\sin\theta} = \frac{\frac{1}{\sin\theta} - \sin\theta}{\sin\theta}, \qquad \textit{using } \csc\theta = \frac{1}{\sin\theta} \tag{96}$$
$$= \frac{\frac{1}{\sin\theta} - \sin\theta}{\sin\theta} \cdot \frac{\sin\theta}{\sin\theta}, \qquad \textit{multiplying top and bottom of the main fraction bar by } \sin\theta \tag{97}$$
$$= \frac{1-\sin^2\theta}{\sin^2\theta}, \qquad \textit{after multiplying by } \sin\theta \tag{98}$$
$$= \frac{\cos^2\theta}{\sin^2\theta}, \qquad \textit{using } \sin^2\theta + \cos^2\theta = 1, \textit{ or } \cos^2\theta = 1 - \sin^2\theta \tag{99}$$
$$= \cot^2\theta_{//}, \qquad \textit{since } \frac{\cos\theta}{\sin\theta} = \cot\theta \tag{100}$$

16. Find all solutions to the following equations: (8 pts)

(a) $\cos\theta\tan\theta - 3\tan\theta = 0$

Solution:

We start by factoring out the $\tan\theta$

$$\cos\theta\tan\theta - 3\tan\theta = 0 \tag{101}$$
$$\tan\theta(\cos\theta - 3) = 0 \tag{102}$$

By the multiplicative property of zero we set $\tan\theta = 0$ and $\cos\theta - 3 = 0$. $\tan\theta = 0$ when $\sin\theta = 0$ which happens at $\theta = 0 + k2\pi$ and $\theta = \pi + k2\pi$ where k is any integer. $\cos\theta - 3 = 0$ is solved if $\cos\theta = 3$ which cannot happen since the largest $\cos\theta$ can be is 1. So the only solutions to the original equation are $\boxed{\theta = 0 + k2\pi \text{ and } \theta = \pi + k2\pi}$ or in simplified form $\boxed{\theta = k\pi}$.

(b) $\sin\left(\frac{\theta}{3}\right) = \frac{1}{2}$

Solution:

$\sin\left(\frac{\theta}{3}\right) = \frac{1}{2}$ is solved when $\frac{\theta}{3} = \frac{\pi}{6} + k2\pi$ and $\frac{\theta}{3} = \frac{5\pi}{6} + k2\pi$ where k is any integer. The resulting solutions for θ occur at $\boxed{\theta = \frac{\pi}{2} + k6\pi \text{ and } \theta = \frac{5\pi}{2} + k6\pi}$.

17. For $m(x) = 2\cos\left(x - \frac{\pi}{4}\right)$ (10 pts)

(a) Identify the amplitude.

Solution:

The amplitude is $|a| = |2| = \boxed{2}$

(b) Identify the period.

Solution:

The period is $\frac{\text{period of cosine}}{|b|} = \frac{2\pi}{|1|} = \boxed{2\pi}$

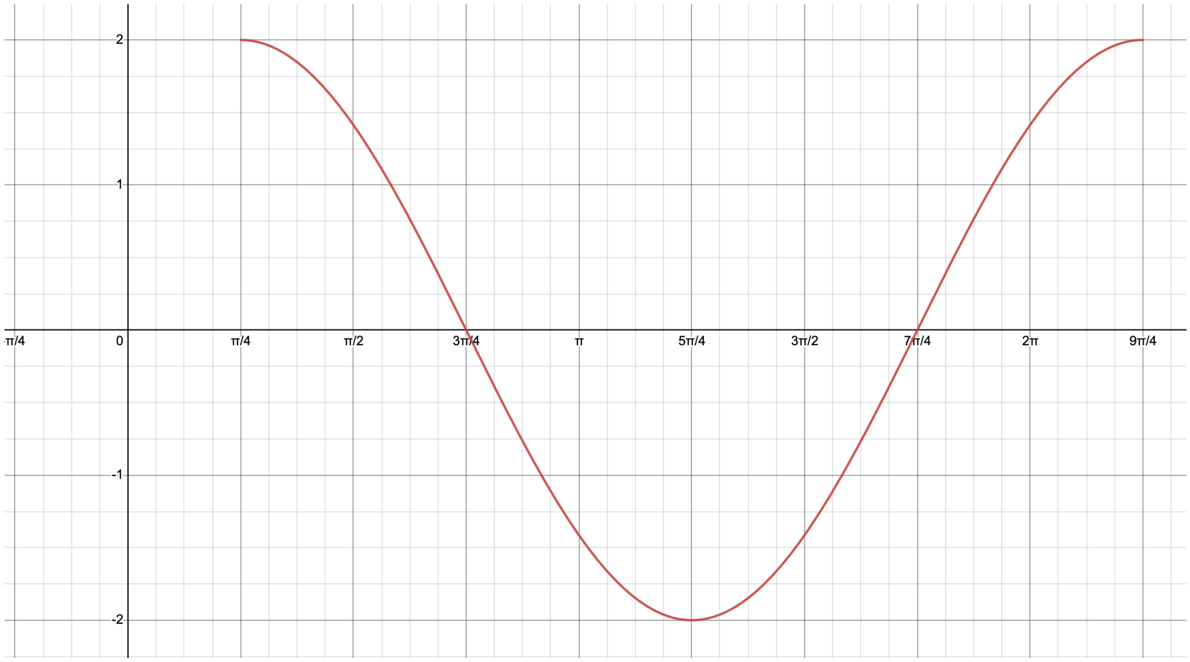
(c) Identify the phase shift.

Solution:

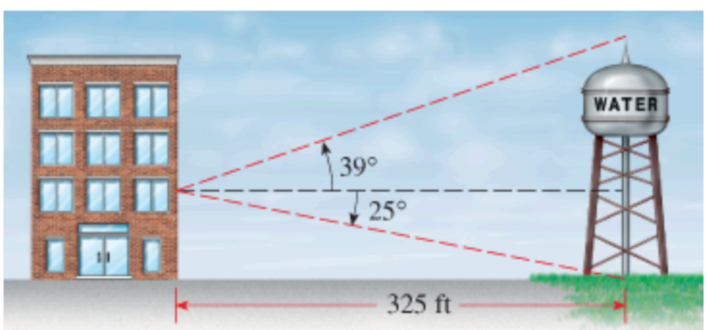
The phase shift is $-\frac{c}{b} = -\frac{-\frac{\pi}{4}}{1} = \boxed{\frac{\pi}{4}}$

(d) Sketch one cycle of the graph of $m(x)$. Label at least two values on the x-axis and clearly identify the amplitude.

Solution:



18. A water tower is located 325 ft from a building. From a window in the building, an observer notes that the angle of elevation to the top of the tower is $39°$ and that the angle of depression to the bottom of the tower is $25°$. **Recall:** You don't have a calculator, so leave answers in exact form. (6 pts)



(a) How high is the **window**?

Solution:

Let's label the side of the right triangle opposite $25°$ to be x. We can then write:

$$\tan(25°) = \frac{x}{325} \tag{103}$$

Solving for x the height of the window is: $\boxed{325\tan(25°)\ \text{ft}}$.

(b) How tall is the **tower**?

Solution:

Each given angle defines a right triangle that we can use trigonometry to work with. Similarly to part (a) we can label the side opposite $25°$ to be x and the side opposite $39°$ to be y. We can then write:

$$\tan(25°) = \frac{x}{325} \tag{104}$$

and

$$\tan(39°) = \frac{y}{325} \tag{105}$$

Solving for x and y respectively, we can write: $325\tan(25°) = x$ and $325\tan(39°) = y$. The height of the tower is $x + y$ so we find the height of the tower to be: $\boxed{325\tan(25°) + 325\tan(39°)\ \text{ft}}$.