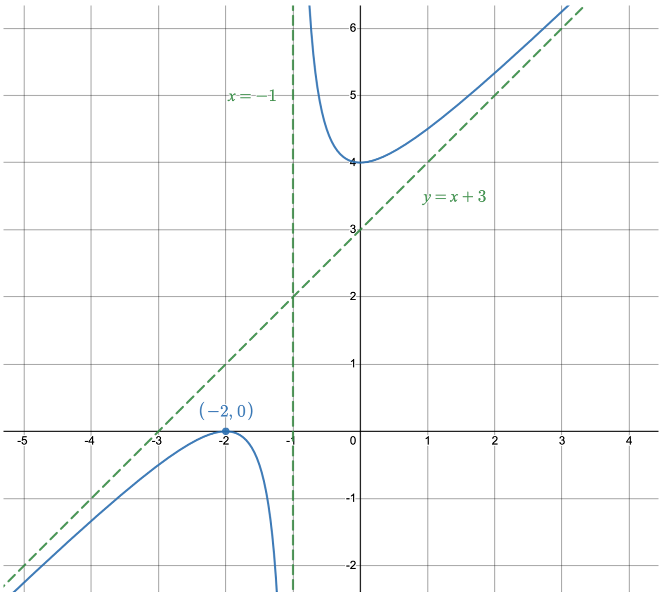
1. Sketch the shape of the graph of a rational function, $g(x)$, that satisfies **all** of the given information. **Label** all intercepts and asymptotes on the graph. (4 pts)

   i. The graph has a slant asymptote: $y = x + 3$

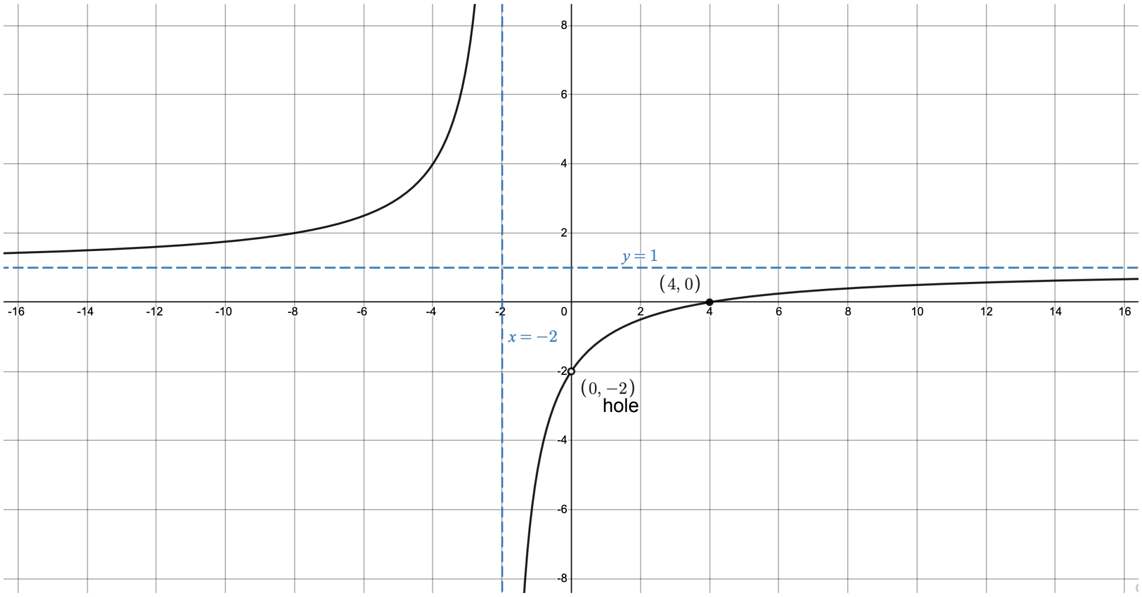
   ii. The graph bounces (touches but does not cross) at x-intercept: $(-2, 0)$

   iii. The graph has no other x-intercepts.

   iv. The graph has a vertical asymptote: $x = -1$.



2. Consider the graph of a rational function below. Write down a rational function, $R(x)$, whose graph has the same vertical asymptote, horizontal asymptote, hole (as labeled), and x-intercept as that of the graph. (4 pts)



Solution:

We start by picking a rational function with x-intercept $(4, 0)$ and vertical asymptote $x = -2$:

$$f(x) = \frac{x-4}{x+2}$$

However this function does not have a hole at $x = 0$. To include a hole, we multiply top and bottom by x:

$$g(x) = \frac{x(x-4)}{x(x+2)}$$

This new function has an x-intercept at $(4, 0)$, vertical asymptote at $x = -2$, and hole at $x = 0$. The last thing to check is the y-coordinate of the hole:

$$\begin{aligned} \frac{0-4}{0+2} &= \frac{-4}{2} && (1) \\ &= -2 && (2) \end{aligned}$$

Which matches the graph. So our final answer is: $\boxed{R(x) = \frac{x(x-4)}{x(x+2)}}$.

3. (a) Simplify (rewrite without logs or exponents): $-\log_3(27) + \log(1) + \ln\left(e^2\right) + 10^0$ (4 pts)

Solution:

$$\begin{aligned} -\log_3(27) + \log(1) + \ln\left(e^2\right) + 10^0 &= -3 + 0 + 2 + 1 && (3) \\ &= \boxed{0} && (4) \end{aligned}$$

(b) Rewrite as a single logarithm: $4\log(x) - \frac{1}{2}\log(y) + \log(z)$ (4 pts)

Solution:

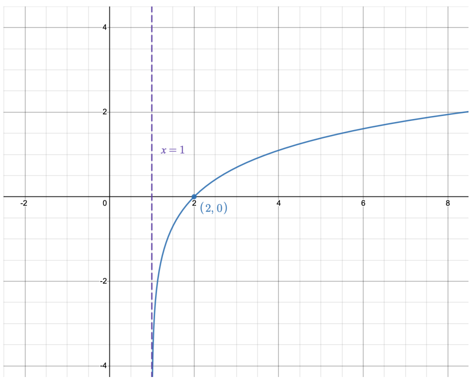
$$4\log(x) - \frac{1}{2}\log(y) + \log(z) = \log\left(x^4\right) - \log\left(y^{1/2}\right) + \log(z) \tag{5}$$
$$= \log\left(\frac{x^4}{y^{1/2}}\right) + \log(z) \tag{6}$$
$$= \boxed{\log\left(\frac{x^4 z}{y^{1/2}}\right)} \tag{7}$$

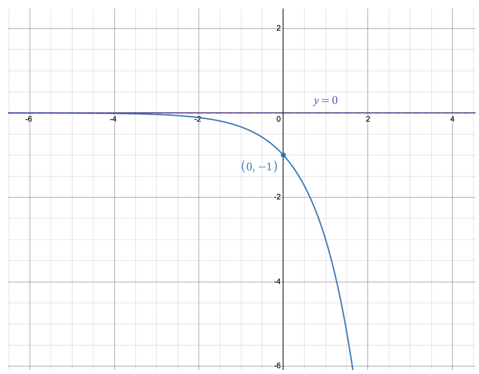
4. For parts (a) and (b) sketch the graphs. **Be sure to label any asymptote(s) and intercept(s) for each graph.** (10 pts)

(a) $f(x) = \ln(x-1)$

(b) $g(x) = -3^x$

Solution:





(c) For the function from part (b) find the value: $g(3\log_3 2)$

Solution:

$$g(3\log_3 2) = -3^{3\log_3 2} \tag{8}$$
$$= -3^{\log_3\left(2^3\right)} \tag{9}$$
$$= -2^3 \tag{10}$$
$$= \boxed{-8} \tag{11}$$

5. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit). (8 pts)

   (a) $10^{x^2-1} = 10^{7-7x}$

   Solution:

   Both sides have the same base of 10, so we can equate exponents by utilizing the one-to-one property for exponential functions and then solve for x:

$$\begin{aligned} x^2 - 1 &= 7 - 7x \quad (12) \\ x^2 + 7x - 8 &= 0 \quad (13) \\ (x+8)(x-1) &= 0 \quad (14) \end{aligned}$$

   So we have two solutions: $\boxed{x = 1, x = -8}$.

   (b) $8 = 2^{-3x-1}$

   Solution:

   One method to solve this is to rewrite 8 as 2^3 and then use the one-to-one property for exponential functions.

$$\begin{aligned} 2^3 &= 2^{-3x-1} \quad (15) \\ 3 &= -3x - 1 \quad (16) \\ 4 &= -3x \quad (17) \\ -\frac{4}{3} &= x \quad (18) \end{aligned}$$

   So we get our answer: $\boxed{x = -\frac{4}{3}}$

6. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit). (8 pts)

   (a) $2x = (x-5)\ln(3)$

   Solution:

   Since $\ln(3)$ is a constant we can optionally make the algebra look more familiar by substituting $c = \ln(3)$:

$$\begin{aligned} 2x &= (x-5)\ln(3) \quad (19) \\ 2x &= (x-5)\,c \quad (20) \\ 2x &= cx - 5c \quad (21) \\ 2x - cx &= -5c \quad (22) \\ x(2-c) &= -5c \quad (23) \\ x &= \frac{-5c}{2-c} \quad (24) \end{aligned}$$

   Replacing $c = \ln(3)$ back in we get our answer $\boxed{x = -\frac{5\ln(3)}{2-\ln(3)}}$

(b) $\log_3(2x-4) - \log_3(2) = \log_3\left(\frac{7}{2}x+1\right)$

Solution:

We start by combining the logarithms on the left so that we can apply the one-to-one property for logarithmic functions:

$$\log_3(2x-4) - \log_3(2) = \log_3\left(\frac{7}{2}x+1\right) \tag{25}$$

$$\log_3\left(\frac{2x-4}{2}\right) = \log_3\left(\frac{7}{2}x+1\right) \tag{26}$$

$$\frac{2x-4}{2} = \frac{7}{2}x+1 \tag{27}$$

$$2x-4 = 7x+2 \tag{28}$$

$$x = -\frac{6}{5} \tag{29}$$

Because this is a logarithmic equation, we need to check if $x = -\frac{6}{5}$ is actually a solution:

On the left hand side we get: $\log_3\left(2\left(-\frac{6}{5}\right)-4\right) - \log_3(2) = \log_3\left(-\frac{32}{5}\right) - \log_3(2)$ which does not exist.

So there are $\boxed{\text{no solutions}}$.

7. Simplify the expression: $\left(7^{2x}\right)^3 + 7^{-x}\left(7^x + 7^{-2x}\right)$ (4 pts)

Solution:

$$\left(7^{2x}\right)^3 + 7^{-x}\left(7^x + 7^{-2x}\right) = 7^{6x} + 7^0 + 7^{-3x} \tag{30}$$

$$= \boxed{7^{6x} + 1 + 7^{-3x}} \tag{31}$$

8. The half-life of cobalt-60 is known to be 5 years. A scientist has a 12 mg (milligram) sample of cobalt-60. (8 pts)

(a) How much of the 12 mg sample of cobalt-60 will remain after 10 years?

Solution:

Half of the original substance remains after 5 years. So after 5 years: $\frac{1}{2} \cdot 12 = 6$ mg remains.

After another 5 years half of 6 mg remains, so after 10 years: $\frac{1}{2} \cdot 6 = \boxed{3 \text{ mg}}$ remains of the original sample.

(b) Find a function $m(t) = m_\circ 2^{-t/h}$ that models the mass remaining after t years.

Solution:

Let $m_\circ = 12$ mg be the initial mass corresponding to $t = 0$. The half-life is 5 years, so $h = 5$ and we have

$\boxed{m(t) = 12 \cdot 2^{-t/5}}$

(c) According to the model, how long will it take until 17% of the initial sample remains? As usual, give your answer in exact form (do not attempt to approximate with a rounded value answer).

Solution:

$m(t)$ represents the mass at any time t. 17% of the initial mass remaining corresponds to $0.17m_\circ$. So we get:

$$m(t) = m_0 \cdot 2^{-t/5} \tag{32}$$
$$0.17m_\circ = m_0 \cdot 2^{-t/5} \tag{33}$$

And solving for t we find:

$$0.17m_\circ = m_0 \cdot 2^{-t/5} \tag{34}$$
$$0.17 = 2^{-t/5} \tag{35}$$
$$\log_2(0.17) = -\frac{t}{5} \tag{36}$$
$$-5\log_2(0.17) = t \tag{37}$$

So the time is: $\boxed{t = -5\log_2(0.17)}$.

9. The following are unrelated. (12 pts)

(a) Determine whether the following two angles are coterminal or not. Make sure to justify your answer to receive credit: $\frac{34\pi}{3}, \frac{20\pi}{3}$

Solution:

To find if the two angles are coterminal, we can subtract 2π from the larger one repeatedly and see if we can equate to the smaller angle.

$$\frac{34\pi}{3} - 2\pi = \frac{34\pi}{3} - \frac{6\pi}{3} \tag{38}$$
$$= \frac{28\pi}{3} \tag{39}$$

We try again:

$$\frac{28\pi}{3} - 2\pi = \frac{28\pi}{3} - \frac{6\pi}{3} \tag{40}$$
$$= \frac{22\pi}{3} \tag{41}$$

If we try one more time we would get $\frac{16\pi}{3}$ which is smaller than $\frac{20\pi}{3}$ so the given angles are $\boxed{\text{not coterminal}}$.

(b) The point, (x, y), lies on the unit circle $x^2 + y^2 = 1$ in quadrant IV. If $x = \frac{2}{\sqrt{7}}$ find y.

Solution:

$$\left(\frac{2}{\sqrt{7}}\right)^2 + y^2 = 1 \tag{42}$$
$$\frac{4}{7} + y^2 = 1 \tag{43}$$
$$y^2 = \frac{3}{7} \tag{44}$$
$$y = \pm\sqrt{\frac{3}{7}} \tag{45}$$

Since we are in quadrant IV, y is negative. Therefore, $\boxed{y = -\sqrt{\frac{3}{7}}}$.

(c) For an angle θ in standard position, suppose we know $\cos(\theta) < 0$ and $\sin(\theta) = -\frac{1}{4}$. What quadrant does the terminal side of θ lie?

Solution:

Since we know that $\cos(\theta) < 0$ is true and $\cos(\theta)$ is the x-coordinate of the point on the unit circle that lies on the terminal side of θ, then we know that the terminal side of θ lies in quadrants II or III.

$\sin(\theta) = -\frac{1}{4}$ means that the y-coordinate of the point on the unit circle that lies on the terminal side of θ is negative which happens when the terminal side of θ lies in quadrants III or IV.

The only quadrant that satisfies both pieces of information is $\boxed{\text{quadrant III}}$.

(d) Simplify: $11\sin^2\left(\frac{\pi}{9}\right) + 11\cos^2\left(\frac{\pi}{9}\right)$

Solution:

We want to apply: $\sin^2(\theta) + \cos^2(\theta) = 1$ but first we need to factor out the 11:

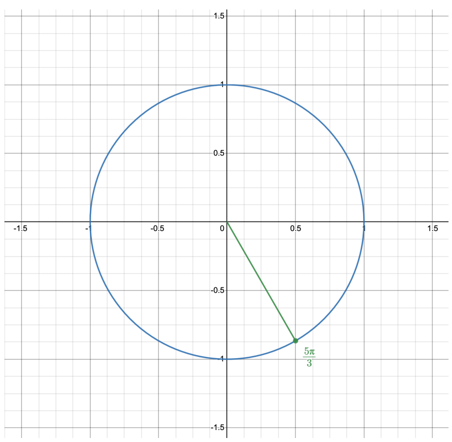
$$\begin{aligned} 11\sin^2\left(\frac{\pi}{9}\right) + 11\cos^2\left(\frac{\pi}{9}\right) &= 11\left(\sin^2\left(\frac{\pi}{9}\right) + \cos^2\left(\frac{\pi}{9}\right)\right) && (46) \\ &= 11(1) && (47) \\ &= \boxed{11} && (48) \end{aligned}$$

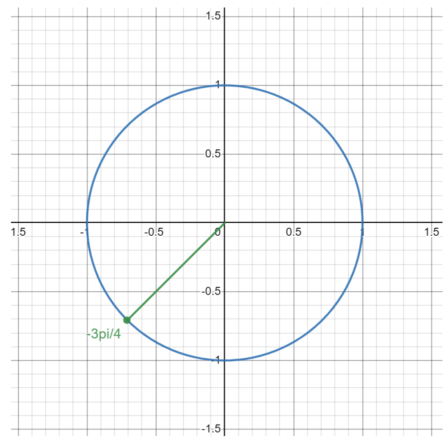
10. Sketch each angle in standard position on the unit circle.

(a) $\theta = \frac{5\pi}{3}$ (2 pts)

(b) $\theta = -\frac{3\pi}{4}$ (2 pts)

Solution:

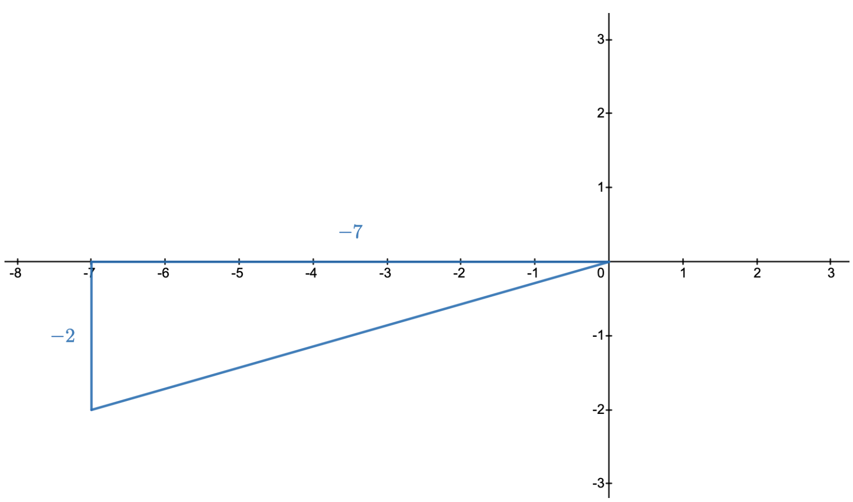
(a) 

(b) 

11. Answer the following for $\tan(\theta) = \frac{2}{7}$ where θ lies in quadrant III.

(a) Sketch a triangle that represents the given information (2 pts).

Solution:



(b) Find $\cot\theta$ (2 pts)

Solution:

We can use the fact that $\cot(\theta) = \dfrac{1}{\tan(\theta)}$ and that we are given the value of $\tan(\theta)$:

$$\begin{align}\cot(\theta) &= \frac{1}{\tan(\theta)} \tag{49}\\ &= \frac{1}{\frac{2}{7}} \tag{50}\\ &= \boxed{\frac{7}{2}} \tag{51}\end{align}$$

(c) Find $\sin\theta$ (3 pts)

Solution:

To find $\sin(\theta)$, we must first find the distance from the origin to the point (we can use Pythagorean identity for this):

$$\begin{align}r^2 &= 2^2 + 7^2 \tag{52}\\ &= 53 \tag{53}\\ &= \sqrt{53} \tag{54}\end{align}$$

Using SOHCAHTOA we find the answer:

$$\boxed{\sin(\theta) = -\frac{2}{\sqrt{53}}}$$

12. Find the exact value of each of the following. If a value does not exist write DNE. (3 pts each)

(a) $\cos(-150°)$

Solution:

$$\cos(-150°) = \boxed{-\frac{\sqrt{3}}{2}}$$

(b) $\sin\left(\frac{\pi}{2}\right)$

Solution:

$$\sin\left(\frac{\pi}{2}\right) = \boxed{1}$$

(c) $\tan\left(\frac{\pi}{6}\right)$

Solution:

$$\tan\left(\frac{\pi}{6}\right) = \boxed{\frac{1}{\sqrt{3}}}$$

(d) $\csc\left(\frac{7\pi}{6}\right)$

Solution:

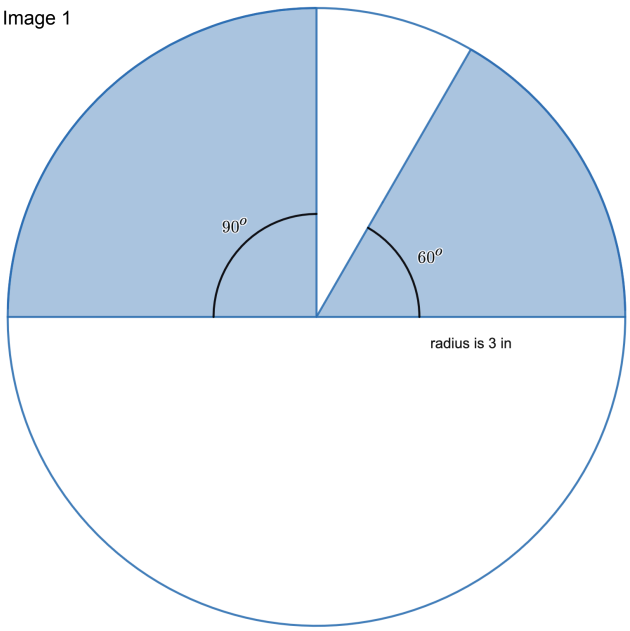
$$\csc\left(\frac{7\pi}{6}\right) = \frac{1}{\sin\left(\frac{7\pi}{6}\right)} = \frac{1}{-\frac{1}{2}} = \boxed{-2}$$

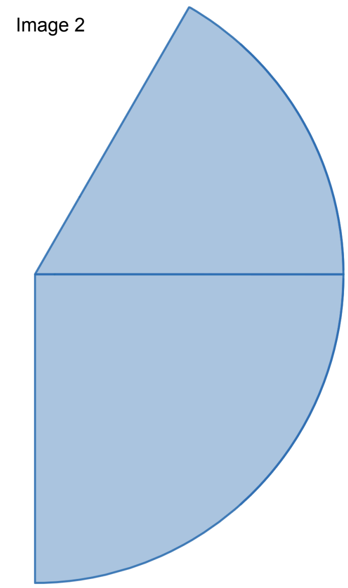
(e) $\sec\left(\frac{3\pi}{4}\right)$

Solution:

$$\sec\left(\frac{3\pi}{4}\right) = \frac{1}{\left(\frac{-1}{\sqrt{2}}\right)} = \boxed{-\sqrt{2}}$$

13. A company is designing a new corporate graphic and they have decided to take two circular sectors (one with central angle $60°$ and the other with central angle $90°$) from inside a circle that has a radius of 3 inches (image 1 below) and reorient them into the new graphic (image 2 below). Find the area of the graphic in image 2. (4 pts)





Solution:

First, we need to convert the given angles into radians: $60° \cdot \dfrac{\pi}{180°} = \dfrac{\pi}{3}$ rad and $90° \cdot \dfrac{\pi}{180°} = \dfrac{\pi}{2}$ rad

Area of the top sector with central angle $60°$ is given by: $A_1 = \dfrac{1}{2}(3^2)\dfrac{\pi}{3} = \dfrac{3\pi}{2}$ in^2

Area of the bottom sector with central angle $90°$ is given by: $A_2 = \dfrac{1}{2}(3^2)\dfrac{\pi}{2} = \dfrac{9\pi}{4}$ in^2

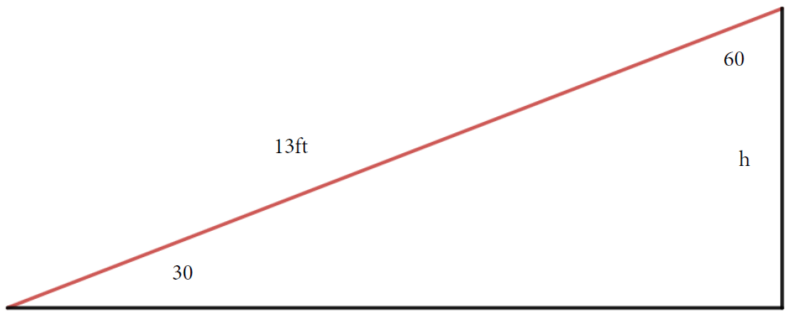
Hence the total area is given by:

$$
\begin{aligned}
A_1 + A_2 &= \frac{3\pi}{2} + \frac{9\pi}{4} \qquad (55)\\
&= \frac{6\pi}{4} + \frac{9\pi}{4} \qquad (56)\\
&= \boxed{\frac{15\pi}{4}\ \text{in}^2} \qquad (57)
\end{aligned}
$$

14. A 13 ft ladder is leaning against a vertical wall such that a triangle is formed between the ground, ladder, and wall. The angle the ladder makes with the wall is $60°$. Find the height of the wall. (4 pts)

Solution:

We start be drawing a diagram and labeling the height the ladder reaches on the wall h.



Using SOHCAHTOA we get:

$$\cos(60°) = \frac{adj}{hyp} = \frac{h}{13}$$

Solving for h we get:

$$h = 13\cos(60°) = \boxed{\frac{13}{2} \text{ ft}}$$

Alternate solution:

We can see that the angle of elevation is $30°$ since the sum of the interior angles of a triangle is $180°$.

$$\sin(30°) = \frac{opp}{hyp} = \frac{h}{13}$$

Solving for h, we obtain

$$h = 13\sin(30°) = \boxed{\frac{13}{2} \text{ ft}}$$