APPM 1235 Final Exam Solutions Fall 2022

INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, electronic devices (such as calculator or other unauthorized electronic resources) are not permitted. **Give all answers in exact form.**

Potentially useful formulas:

1. $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
2. Circle: $(x-h)^2 + (y-k)^2 = r^2$
3. Arc length: $s = r\theta$
4. $\sin(a-b) = \sin a \cos b - \sin b \cos a$
5. $\cos(a-b) = \cos a \cos b + \sin a \sin b$
6. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
7. $\cos(2\theta) = 2\cos^2\theta - 1$
8. $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$
9. $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
10. $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
11. Area of a sector: $A = \frac{1}{2}r^2\theta$
12. $\sin(a+b) = \sin a \cos b + \sin b \cos a$
13. $\cos(a+b) = \cos a \cos b - \sin a \sin b$
14. $\sin(2\theta) = 2\sin\theta\cos\theta$
15. $\cos(2\theta) = 1 - 2\sin^2\theta$
16. $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$
17. $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

NOTE: YOU MAY TEAR OFF THIS FIRST PAGE AND THE NEXT PAGE AND USE (FRONT AND BACK) OF BOTH AS SCRATCH PAPER.

i. DO NOT START UNTIL INSTRUCTED BY A PROCTOR.

ii. THE EXAM IS ON BOTH SIDES OF EACH EXAM PAGE

iii. WRITE YOUR NAME ON THE FIRST EXAM PAGE.

iv. WHEN YOU FINISH (IF BEFORE THE EXAM END TIME) PLEASE QUIETLY COLLECT YOUR THINGS AND EXIT TO THE EXAM SUBMISSION AREA.

Name: ________________________________ Upload time: __________________

1. The following are unrelated. Simplify answers and leave without negative exponents. (16 pts)

   (a) Add/subtract as indicated: $\sqrt{24}+\frac{1}{2}-\frac{5}{6}-\frac{\frac{8}{3}}{2}$

   Solution:

$$\begin{aligned}
\sqrt{24}+\frac{1}{2}-\frac{5}{6}-\frac{8}{6} &= \frac{12\sqrt{6}}{6}+\frac{3}{6}-\frac{5}{6}-\frac{8}{6} && (1)\\
&= \frac{12\sqrt{6}+3-5-8}{6} && (2)\\
&= \frac{12\sqrt{6}-10}{6} && (3)\\
&= \boxed{\frac{6\sqrt{6}-5}{3}} && (4)
\end{aligned}$$

   (b) Simplify: $-\left(x^2-1\right)^2-\left(-x^3\right)^2\left(\frac{x^{-3}}{x}\right)$

   Solution:

$$\begin{aligned}
&= -(x^4-2x^2+1)-x^6\cdot\frac{1}{x^4} && (5)\\
&= -x^4+2x^2-1-x^2 && (6)\\
&= \boxed{-x^4+x^2-1} && (7)
\end{aligned}$$

   (c) Multiply and simplify: $\frac{x^2+x+1}{x^3-16x}\cdot\frac{x-4}{x^3-1}$

   Solution:

$$\begin{aligned}
&= \frac{x^2+x+1}{x(x^2-16)}\cdot\frac{x-4}{(x-1)(x^2+x+1)} && (8)\\
&= \frac{x^2+x+1}{x(x+4)(x-4)}\cdot\frac{x-4}{(x-1)(x^2+x+1)} && (9)\\
&= \boxed{\frac{1}{x(x+4)(x-1)}} && (10)
\end{aligned}$$

   (d) Simplify: $\sqrt{x}\left(\sqrt{x}-x^{2/3}\right)+\frac{1}{\left(x^{-1/9}\right)^3}$

   Solution:

$$\begin{aligned}
&= x^{\frac{1}{2}+\frac{1}{2}}-x^{\frac{2}{3}+\frac{1}{2}}+\frac{1}{x^{-\frac{1}{3}}} && (11)\\
&= \boxed{x-x^{\frac{7}{6}}+x^{\frac{1}{3}}} && (12)
\end{aligned}$$

2. The following are unrelated. Simplify answers and leave without negative exponents. (8 pts)

   (a) Simplify: $\dfrac{\frac{3}{x}+\frac{3}{x^2}}{\frac{1}{x}+1}$

   Solution:

$$= \frac{\frac{3x}{x^2}+\frac{3}{x^2}}{\frac{1}{x}+\frac{x}{x}} \tag{13}$$
$$= \frac{\frac{3x+3}{x^2}}{\frac{1+x}{x}} \tag{14}$$
$$= \frac{3x+x}{x^2}\cdot\frac{x}{1+x} \tag{15}$$
$$= \frac{3(x+1)x}{x^2(1+x)} \tag{16}$$
$$= \boxed{\frac{3}{x}} \tag{17}$$

   (b) Simplify: $\log_4(64) - \log_5\left(5^3\right) + \log_4(32) - \log_4(2) + \log(1)$ (Your answer should have no logarithms)

   Solution:

   Solving for one logarithm at a time:
   $\log_4(64) = 3$
   $\log_5(5^3) = 3$
   $\log_4(32) \implies 4^x = 32 \implies 2^{2x} = 2^5$ thus $\log_4(32) = \frac{5}{2}$
   $\log_4(2) = \frac{1}{2}$
   $\log(1) = 0$
   Putting that all together:

$$\log_4(64) - \log_5\left(5^3\right) + \log_4(32) - \log_4(2) + \log(1) = 3 - 3 + \frac{5}{2} - \frac{1}{2} + 0 \tag{18}$$
$$= \boxed{2} \tag{19}$$

3. Find all real and complex solutions: $x^3 - x^2 + x = 0$. Give complex solutions in $a + bi$ form. (4 pts)

   Solution:

$$x(x^2 - x + 1) = 0 \tag{20}$$
$$x = 0, \quad x^2 - x + 1 = 0 \tag{21}$$

   One solution will be $x = 0$, and the others will occur when $x^2 - x + 1 = 0$. Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1-4}}{2} \tag{22}$$
$$= \frac{1 \pm \sqrt{-3}}{2} \tag{23}$$
$$= \frac{1 \pm \sqrt{3}i}{2} \tag{24}$$

   Therefore, the solutions are $\boxed{x = 0 + 0i}$, $\boxed{x = \frac{1}{2} + \frac{\sqrt{3}}{2}i}$, and $\boxed{x = \frac{1}{2} - \frac{\sqrt{3}}{2}i}$.

4. Solve the following equations for the indicated variable. If there are no solutions, write **no solutions**. (16 pts)

(a) Solve for n: $\dfrac{3}{n+1} = \dfrac{1}{2n}$

Solution:

$$\begin{aligned}
\frac{3}{n+1} &= \frac{1}{2n} && (25)\\
6n &= n+1 && (26)\\
5n &= 1 && (27)\\
n &= \boxed{\frac{1}{5}} && (28)
\end{aligned}$$

(b) Solve for r: $U - U_\circ = \dfrac{kq_1q_2}{r^2}$

Solution:

$$\begin{aligned}
U - U_\circ &= \frac{kq_1q_2}{r^2} && (29)\\
r^2(U - U_\circ) &= kq_1q_2 && (30)\\
r^2 &= \frac{kq_1q_2}{U - U_\circ} && (31)\\
r &= \boxed{\pm\sqrt{\frac{kq_1q_2}{U - U_\circ}}} && (32)
\end{aligned}$$

(c) Solve for x: $3 + 2\log_4(x) = 1$

Solution:

$$\begin{aligned}
3 + 2\log_4(x) &= 1 && (33)\\
2\log_4(x) &= -2 && (34)\\
\log_4(x) &= -1 && (35)\\
4^{-1} &= x && (36)\\
x &= \boxed{\frac{1}{4}} && (37)
\end{aligned}$$

(d) Solve for t: $7e^{-0.2t} = 12$

Solution:

$$\begin{aligned}
7e^{-0.2t} &= 12 && (38)\\
e^{-0.2t} &= \frac{12}{7} && (39)\\
-0.2t &= \ln\left(\frac{12}{7}\right) && (40)\\
t &= -\frac{1}{0.2}\ln\left(\frac{12}{7}\right) && (41)\\
&= \boxed{-5\ln\left(\frac{12}{7}\right)} && (42)
\end{aligned}$$

5. For the two points $P(-3,1)$ and $Q(-1,4)$ find: (11 pts)

(a) The slope of the line through the two points.

Solution:

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{4-1}{-1+3} = \boxed{\frac{3}{2}} \tag{43}$$

(b) The equation of the line that passes through the points P and Q.

Solution:

$$y - y_1 = m(x - x_1) \tag{44}$$
$$y - 4 = \frac{3}{2}(x+1) \tag{45}$$
$$y = \boxed{\frac{3}{2}x + \frac{11}{2}} \tag{46}$$

(c) Find the equation of the circle centered at Q that passes through P.

Solution:

The center of the circle is $Q(-1,4)$. For $P(-3,1)$ to be on the circle, the radius of the circle must be the distance between P and Q. That is,

$$r = \sqrt{(4-1)^2 + (-1+3)^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

The equation of the circle is then $\boxed{(x+1)^2 + (y-4)^2 = 13}$.

6. Consider the functions: $m(x) = \sqrt{x-3}$, $r(x) = x^2 - 2$, and $q(x) = e^{2x}$. (12 pts)

(a) Find the domain of $m(x)$. Give your answer in interval notation.

Solution:

The domain of $m(x)$ includes values of x that make $\sqrt{x-3}$ a real number. Thus, we need

$$x - 3 \geq 0 \tag{47}$$
$$x \geq 3 \tag{48}$$

In interval notation, the domain is $\boxed{[3, \infty)}$.

(b) Find $(r \circ m)(x)$.

Solution:
$(r \circ m)(x) = r(m(x)) = r(\sqrt{x-3}) = (\sqrt{x-3})^2 - 2 = x - 3 - 2 = \boxed{x-5}$

(c) Find the domain of $(r \circ m)(x)$. Give your answer in interval notation.

Solution:

Since the domain of $m(x)$ is $[3, \infty)$, the domain of $(r \circ m)(x)$ is restricted to $[3, \infty)$. The domain of $r(x)$ is all real numbers, which does not restrict the domain of $(r \circ m)(x)$ any further, so the final answer is $\boxed{[3, \infty)}$.

(d) Find $q(\ln(2))$.

Solution:

$q(\ln(2)) = e^{2\ln(2)} = e^{\ln(2^2)} = 2^2 = \boxed{4}$

7. For $P(x) = x^4 - x^3 - 6x^2$ answer the following (13 pts):

(a) Find the x-intercept(s).

Solution:

Setting $y = P(x) = 0$ yields

$$x^4 - x^3 - 6x^2 = 0 \tag{49}$$
$$x^2(x^2 - x - 6) = 0 \tag{50}$$
$$x^2(x-3)(x+2) = 0 \tag{51}$$
$$x = -2, 0, 3 \tag{52}$$

The x-intercepts are then $\boxed{(-2, 0)}$, $\boxed{(0, 0)}$, and $\boxed{(3, 0)}$.

(b) Find the y-intercept.

Solution:

Setting $x = 0$, gives $P(0) = 0 - 0 - 0 = 0$. The y-intercept is then $\boxed{(0, 0)}$.

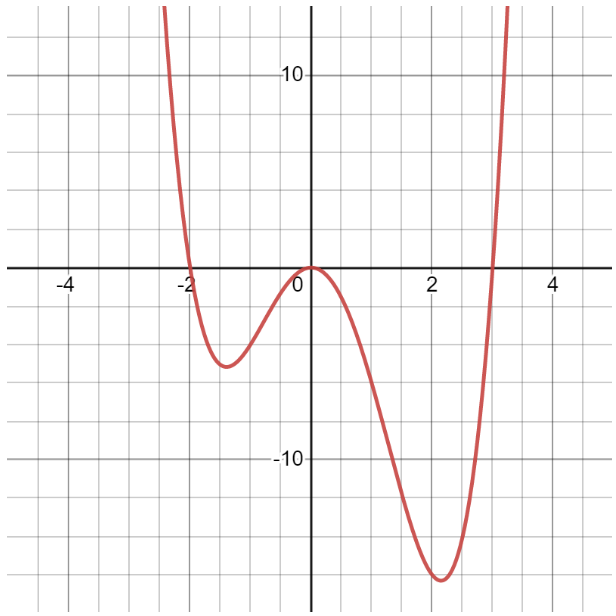
(c) Indicate on a graph or use arrow notation to indicate the end behavior of $P(x)$.

Solution:

The leading term, x^4, has an even exponent and a positive coefficient, so $\boxed{P(x) \to \infty \; as \; x \to \infty}$ and $\boxed{P(x) \to \infty \; as \; x \to -\infty}$.

(d) Graph $P(x)$.

Solution:



8. Sketch the graph of $y = R(x)$ with the following properties: (5 pts)

   i. The graph of $y = R(x)$ has a slant asymptote of $y = x - 1$.

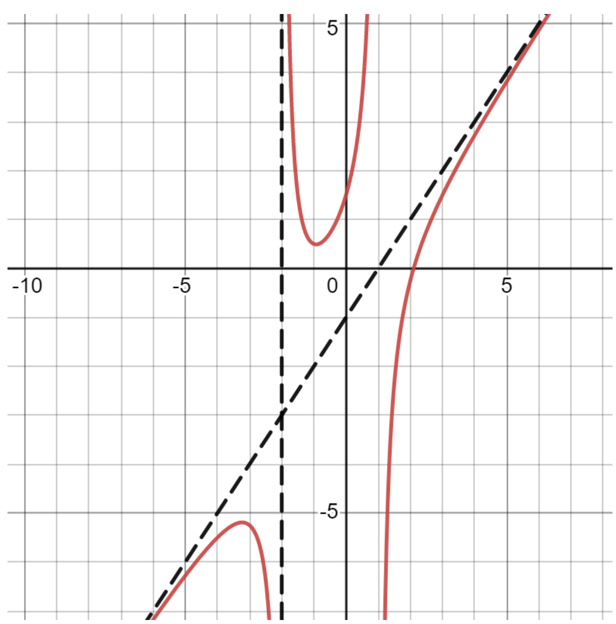
   ii. The graph of $y = R(x)$ crosses the x-intercept $(2, 0)$.

   iii. The graph of $y = R(x)$ has vertical asymptote $x = -2$.

   iv. The graph has no other x-intercepts.

Solution:

Note that there are many possible solutions. This is just one.

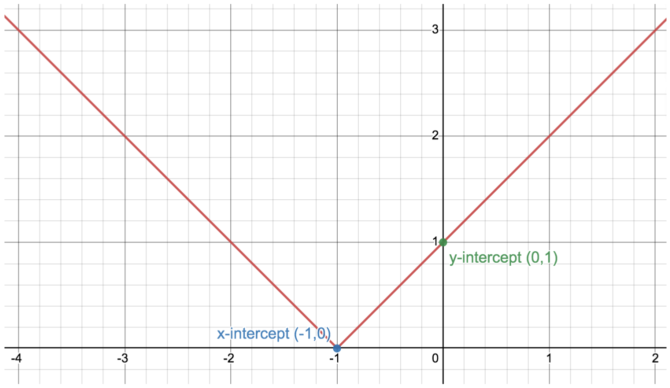


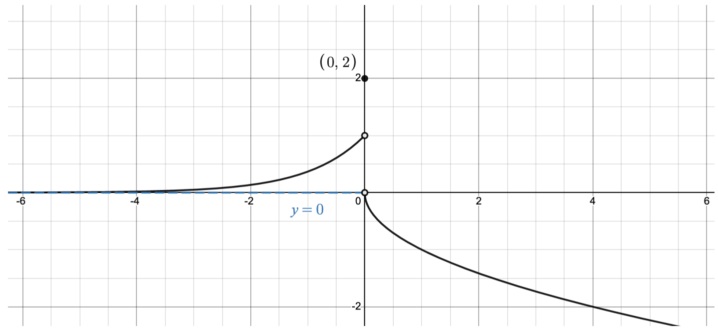
9. Sketch the graph of the following functions. Label all intercepts and asymptotes as appropriate. (14 pts)

(a) $f(x) = |x + 1|$.

(c) $q(x) = \begin{cases} e^x & \text{if} \quad x < 0 \\ 2 & \text{if} \quad x = 0 \\ -\sqrt{x} & \text{if} \quad x > 0 \end{cases}$

Solution:

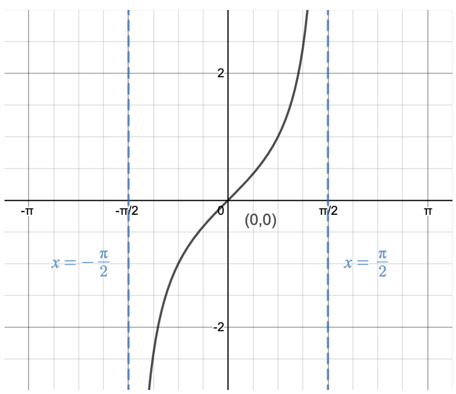


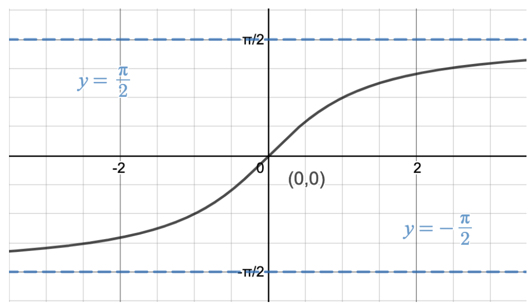


(b) $h(x) = \tan(x)$ on the restricted domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(d) $k(x) = \tan^{-1}(x)$

Solution:





10. Find the exact value: (18 pts)

(a) $\sin\left(\frac{2\pi}{3}\right)$

Solution:

$$\sin\left(\frac{2\pi}{3}\right) = \boxed{\frac{\sqrt{3}}{2}}$$

(b) $\cos\left(-\frac{\pi}{4}\right)$

Solution:

$$\cos\left(-\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$$

(c) $\arctan\left(\dfrac{1}{\sqrt{3}}\right)$

Solution:

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}}$$

(d) $\cos\left(\sin^{-1}(1)\right)$

Solution:

$$\cos\left(\sin^{-1}(1)\right) = \boxed{0}$$

(e) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$

Solution:

$$\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) = \boxed{\frac{\pi}{6}}$$

(f) $\cos\left(\dfrac{3\pi}{8}\right)$

Solution:

$$\cos\left(\frac{3\pi}{8}\right) = \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

11. Verify the identity: $\dfrac{\sec(\theta)\cos^2(\theta)+\sin(\theta)\tan(\theta)}{\sec(\theta)} = 1$. (5 pts)

Solution:

This is one possible solution:

$$\begin{aligned}
LHS &= \frac{\sec(\theta)\cos^2(\theta)+\sin(\theta)\tan(\theta)}{\sec(\theta)} && (53)\\
&= \frac{\sec(\theta)\cos^2(\theta)+\sin(\theta)*\frac{\sin(\theta)}{\cos(\theta)}}{\sec(\theta)} && (54)\\
&= \frac{\sec(\theta)\cos^2(\theta)+\sin^2(\theta)*\frac{1}{\cos(\theta)}}{\sec(\theta)} && (55)\\
&= \frac{\sec(\theta)\cos^2(\theta)+\sin^2(\theta)\sec(\theta)}{\sec(\theta)} && (56)\\
&= \frac{\sec(\theta)\left(\cos^2(\theta)+\sin^2(\theta)\right)}{\sec(\theta)} && (57)\\
&= \cos^2(\theta)+\sin^2(\theta) && (58)\\
&= 1 = RHS && (59)
\end{aligned}$$

12. Is $f(x) = \sin^2(x)\cos(x)$ odd, even, or neither? Justify your answer for credit. (4 pts)

Solution:

We want to test both $f(x) = f(-x)$ and $f(-x) = -f(x)$ (which test even and oddness respectively). If neither statement is true, then the function is neither even nor odd.

Let's first start with the even property.

$$\begin{aligned} f(-x) &= \sin^2(-x)\cos(-x) && (60)\\ &= (\sin(-x))^2\cos(x) && (61)\\ &= (-\sin(x))^2\cos(x) && (62)\\ &= \sin^2(x)\cos(x) && (63)\\ &= f(x) && (64) \end{aligned}$$

Thus, $f(x) = \sin^2(x)\cos(x)$ is $\boxed{\text{even}}$ as a result.

13. Find all solutions to the following equations: (8 pts)

(a) $\sin(\theta) - \tan(\theta)\sin(\theta) = 0$

Solution:

By factoring we get:

$$\begin{aligned} \sin(\theta) - \tan(\theta)\sin(\theta) &= 0 && (65)\\ \sin(\theta)\,(1 - \tan(\theta)) &= 0 && (66) \end{aligned}$$

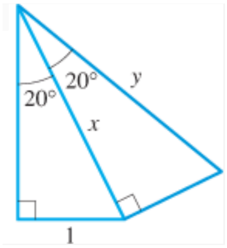
and by using the multiplicative property of zero we get $\sin(\theta) = 0$ and $1 - \tan(\theta) = 0$. $\sin(\theta) = 0$ has solutions $\theta = 0 + k2\pi$ and $\theta = \pi + k2\pi$ where k is any integer. These can by combined into a simplified form: $\theta = k\pi$. $1 - \tan(\theta) = 0$ has solutions when $\tan(\theta) = 1$ which occurs when $\theta = \dfrac{\pi}{4} + k2\pi$ and $\theta = \dfrac{5\pi}{4} + k2\pi$. These can also be combined into a simplified form: $\theta = \dfrac{\pi}{4} + k\pi$. The resulting solutions of the original equation are: $\boxed{\theta = k\pi}$ and $\boxed{\theta = \dfrac{\pi}{4} + k\pi}$.

(b) $\cos\left(\dfrac{\theta}{3}\right) = \dfrac{1}{2}$

Solution:

The equation, $\cos\left(\dfrac{\theta}{3}\right) = \dfrac{1}{2}$, is solved when $\dfrac{\theta}{3} = \dfrac{\pi}{3} + k2\pi$ and $\dfrac{\theta}{3} = \dfrac{5\pi}{3} + k2\pi$ where k is any integer. Multiplying both sides by 3 we find the solutions of the original equation are: $\boxed{\theta = \pi + k6\pi}$ and $\boxed{\theta = 5\pi + k6\pi}$.

14. Solve for both x and y. (6 pts)



Solution:

Starting with the right triangle involving the side length of 1, we can write down: $\sin(20°) = \frac{1}{x}$. Solving for x we get $x = \frac{1}{\sin(20°)}$. For the second triangle we can write down: $\cos(20°) = \frac{x}{y}$ and solving for y we get $y = \frac{x}{\cos(20°)}$ substituting in for x, we get $y = \frac{\frac{1}{\sin(20°)}}{\cos(20°)} = \frac{1}{\sin(20°)\cos(20°)}$.

15. For $f(x) = 2\sin\left(x - \frac{\pi}{3}\right)$ (10 pts)

(a) Identify the amplitude.

Solution:

Amplitude is $|a| = 2$.

(b) Identify the period.

Solution:

Period is $\frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$.

(c) Identify the horizontal shift.

Solution:

Horizontal shift is $-\frac{-\frac{\pi}{3}}{b} = -\frac{-\frac{\pi}{3}}{1} = \frac{\pi}{3}$.

(d) Sketch one cycle of the graph of $f(x)$. Be sure to label at least two values on the x axis and clearly identify the amplitude.