INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, electronic devices (such as calculator or other unauthorized electronic resources) are not permitted. **Give all answers in exact form.**

Potentially useful formulas:

$$\log_b(B) = \frac{\log_a(B)}{\log_a(b)} \text{ for } a > 0,\, a \neq 1.$$

$$A = \frac{1}{2}r^2\theta$$

$$S = r\theta$$

1. For $R(x) = \dfrac{x^2 - 16}{2x^2 - 2x - 24}$ (9 pts)

   (a) Find the location (x, y -coordinates) of any hole(s). If there are none state NONE.

   Solution:

   Factor the numerator and denominator to see if any terms cancel:

   $$\frac{x^2 - 16}{2x^2 - 2x - 24} = \frac{(x+4)(x-4)}{2(x-4)(x+3)}$$

   Because $(x-4)$ is both in the numerator and denominator, they can cancel and the hole is at $x = 4$.

   To find the y-coordinate of the hole, we plug in the x value into the simplified version of $R(x)$:

   $$\frac{(x+4)(x-4)}{2(x-4)(x+3)} = \frac{x+4}{2(x+3)}$$

   $$\text{plugging in x = 4: } \frac{4+4}{2(4+3)} = \frac{4}{7}$$

   Therefore, the hole is at $\boxed{\left(4, \frac{4}{7}\right)}$.

   Note: Because the rational function has a hole in it, we use the simplified version of the function to find vertical and horizontal asymptotes.

   (b) Find any horizontal or slant asymptote(s). If there are none state NONE.

   Solution:

   Horizontal asymptotes occur when the power of the numerator is the same or smaller than the power of the denominator. Because the power of the numerator is 1 and the power of the denominator is 1, we find the horizontal asymptote by dividing the term of the largest power in the numerator by the term of the largest power in the denominator:

   $$\frac{x}{2x} = \frac{1}{2}$$

   Therefore, the horizontal asymptote is at $\boxed{\mathbf{y = \frac{1}{2}}}$, and there are no slant asymptotes.

(c) Find any vertical asymptote(s). If there are none state NONE.

Solution:

Vertical asymptotes occur wherever the denominator of a rational function equals zero. Thus, we set the denominator equal to zero and solve:

$$\begin{aligned} 2(x+3) &= 0 \\ x &= -3 \end{aligned}$$

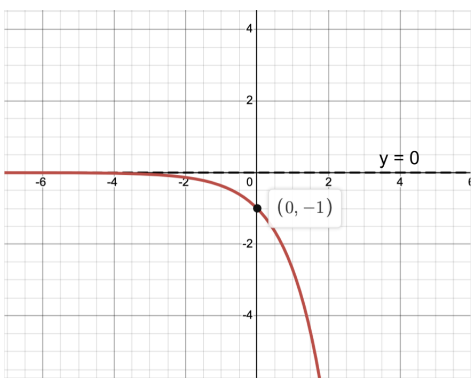
Thus, the vertical asymptote is $\boxed{x = -3}$.

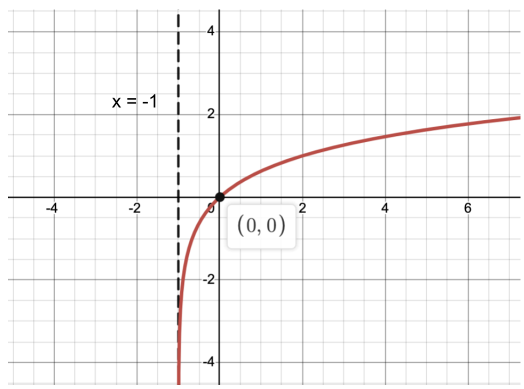
2. Sketch the following graphs: Be sure to label any asymptotes and intercepts for each graph.

(a) $q(x) = -e^x$ (3 pts)

(b) $f(x) = \log_3(x+1)$ (3 pts)

Solution:





3. Simplify: $2^x\left(2^x - 2^{-x}\right) - (2^x)^2$. (3 pts)

Solution:

$$\begin{aligned} 2^x\left(2^x - 2^{-x}\right) - (2^x)^2 &= 2^{2x} - 2^0 - 2^{2x} \\ &= 2^{2x} - 1 - 2^{2x} \\ &= \boxed{-1} \end{aligned}$$

4. (a) Simplify (rewrite without logs): $-\log_3(27) + \log_9(9^{\sqrt{2}}) + \log_5\left(\frac{1}{\sqrt{5}}\right) - e^{\ln(3)}$ (4 pts)

Solution:

$$\begin{aligned} -\log_3(27) + \log_9(9^{\sqrt{2}}) + \log_5\left(\frac{1}{\sqrt{5}}\right) - e^{\ln(3)} &= -\log_3(3^3) + \log_9(9^{\sqrt{2}}) + \log_5\left(5^{-1/2}\right) - 3 \\ &= -3\log_3(3) + \sqrt{2}\log_9(9) - \frac{1}{2}\log_5(5) - 3 \\ &= -3 + \sqrt{2} - \frac{1}{2} - 3 \\ &= \boxed{-\frac{13}{2} + \sqrt{2}} \end{aligned}$$

(b) Rewrite as a single logarithm without negative exponents: $3\ln\left(z^2\right) - 4\ln(z) - \frac{2}{3}\ln(z)$ (4 pts)

Solution:

$$\begin{aligned}
3\ln\left(z^2\right) - 4\ln(z) - \frac{2}{3}\ln(z) &= \ln\left(z^6\right) - \ln\left(z^4\right) - \ln\left(z^{2/3}\right) \\
&= \ln\left(\frac{z^6}{z^4}\right) - \ln\left(z^{2/3}\right) \\
&= \ln\left(z^2\right) - \ln\left(z^{2/3}\right) \\
&= \ln\left(\frac{z^2}{z^{2/3}}\right) \\
&= \boxed{\ln\left(z^{4/3}\right)}
\end{aligned}$$

5. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit).

(a) $10 = 4e^{-2x}$ (4 pts)

Solution:

$$\begin{aligned}
10 &= 4e^{-2x} \\
\frac{5}{2} &= e^{-2x} \\
\ln\left(\frac{5}{2}\right) &= -2x \\
-\frac{1}{2}\ln\left(\frac{5}{2}\right) &= x
\end{aligned}$$

So, $\boxed{x = -\frac{1}{2}\ln\left(\frac{5}{2}\right)}$.

(b) $7 = 2^{3+2x}$ (4 pts)

Solution:

$$\begin{aligned}
7 &= 2^{3+2x} \\
\log_2(7) &= \log_2\left(2^{3+2x}\right) \\
\log_2(7) &= 3 + 2x \\
\log_2(7) - 3 &= 2x \\
\frac{1}{2}\log_2(7) - \frac{3}{2} &= x
\end{aligned}$$

Thus $\boxed{x = \frac{1}{2}\log_2(7) - \frac{3}{2}}$. Note that there are multiple solutions to this problem as the base of the log can vary.

6. Solve the following equations for x. If there are no solutions write "no solutions" (be sure to justify answer for full credit).

(a) $3^{5x+1} = 9^x$ (4 pts)

Solution:

$$\begin{aligned}
3^{5x+1} &= 9^x \\
3^{5x+1} &= \left(3^2\right)^x \\
3^{5x+1} &= 3^{2x} \\
5x+1 &= 2x \\
3x &= -1 \\
x &= -\frac{1}{3}
\end{aligned}$$

Thus $\boxed{x = -\frac{1}{3}}$.

(b) $\log(2x) = \log\left(3x^2 - 9\right) - \log(3)$ (4 pts)

Solution:

$$\begin{aligned}
\log(2x) &= \log\left(3x^2 - 9\right) - \log(3) \\
\log(2x) &= \log\left(\frac{3x^2 - 9}{3}\right) \\
\log(2x) &= \log\left(x^2 - 3\right) \\
2x &= x^2 - 3 \\
0 &= x^2 - 2x - 3 \\
0 &= (x-3)(x+1)
\end{aligned}$$

Thus $x = 3$ and $x = -1$, but we have to see if these satisfy the original equation.

For $x = -1$, we get

$$\begin{aligned}
\log(2(-1)) &\stackrel{?}{=} \log\left(3(-1)^2 - 9\right) - \log(3) \\
\log(-2) &\stackrel{?}{=} \log(-6) - \log(3) \\
x = -1 & \text{ cannot be a solution as this results in a negative inside the logarithms.}
\end{aligned}$$

For $x = 3$, we get

$$\begin{aligned}
\log(2(3)) &\stackrel{?}{=} \log\left(3(3)^2 - 9\right) - \log(3) \\
\log(6) &\stackrel{?}{=} \log(18) - \log(3) \\
\log(6) &\stackrel{?}{=} \log\left(\frac{18}{3}\right) \\
\log(6) &\stackrel{?}{=} \log(6)
\end{aligned}$$

$\log(6) = \log(6)$, so the only solution to the above equation is $\boxed{x = 3}$.

(c) $5 = 2\log_2\left(x - \frac{1}{2}\right)$ (4 pts)

Solution:

$$\begin{aligned}
5 &= 2\log_2\left(x - \frac{1}{2}\right) \\
\frac{5}{2} &= \log_2\left(x - \frac{1}{2}\right) \\
2^{5/2} &= 2^{\log_2\left(x-\frac{1}{2}\right)} \\
\sqrt{32} &= x - \frac{1}{2} \\
\sqrt{16\cdot 2} &= x - \frac{1}{2} \\
4\sqrt{2} &= x - \frac{1}{2} \\
4\sqrt{2} + \frac{1}{2} &= x
\end{aligned}$$

We check to see if our potential solutions solves the original equation:

$$\begin{aligned}
5 &\stackrel{?}{=} 2\log_2\left(4\sqrt{2} + \frac{1}{2} - \frac{1}{2}\right) \\
5 &\stackrel{?}{=} 2\log_2\left(4\sqrt{2}\right) \\
5 &\stackrel{?}{=} 2\log_2(4) + 2\log_2(\sqrt{2}) \\
5 &\stackrel{?}{=} 2\log_2(2^2) + 2\log_2(2^{1/2}) \\
5 &\stackrel{?}{=} 4\log_2(2) + 2 * \frac{1}{2}\log_2(2) \\
5 &\stackrel{?}{=} 4 + 1
\end{aligned}$$

$5 = 5$, thus $\boxed{x = 4\sqrt{2} + \frac{1}{2}}$ is our solution.

7. The raccoon population in a certain Colorado town was 1000 in 2022, and the population is expected to double every 5 years.

(a) Find an exponential model $n(t) = n_\circ 2^{t/a}$ for the number of raccoons after t years. (3 pts)

Solution:

The initial population in 2022 when we say $t = 0$ is 1000. That is, $n(0) = 1000$. Therefore,

$$1000 = n(0) = n_\circ 2^{0/a} = n_\circ,$$

so $n_\circ = 1000$. Since the population is expected to double every 5 years and $n(0) = 1000$, then $n(5) = 2000$. Therefore,

$$\begin{aligned}
2000 &= n_\circ 2^{t/a} = 1000 \cdot 2^{5/a} \\
2 &= 2^{5/a} \\
\log_2 2 &= \log_2 2^{5/a} \\
1 &= \frac{5}{a} \\
a &= 5
\end{aligned}$$

Therefore, the model is $\boxed{n(t) = 1000 \cdot 2^{t/5}}$

(b) What is the expected population size in 2032? (3 pts)

Solution:

If $t = 0$ represents 2022, then $t = 10$ represents 2032. The expected population size in 2032 is therefore $n(10) = 1000 \cdot 2^{10/5} = 1000 \cdot 2^2 = \boxed{4000 \text{ raccoons}}$.

(c) How many years until the population triples? (3 pts)

Solution:

The population is tripled when $n(t) = 3 \cdot 1000 = 3000$. To find when this happens, we solve for t:

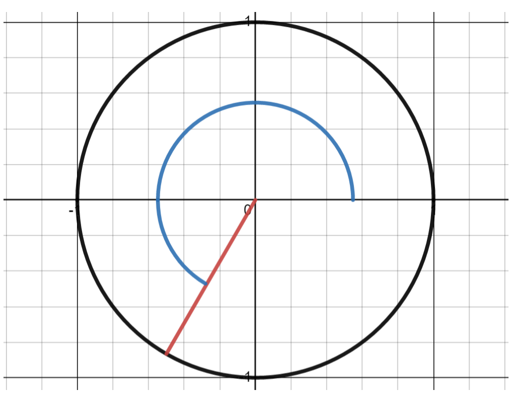
$$\begin{aligned} n(t) = 1000 \cdot 2^{t/5} &= 3000 \\ 2^{t/5} &= 3 \\ \log_2 2^{t/5} &= \log_2 3 \\ \frac{t}{5} &= \log_2 3 \\ t &= 5 \log_2 3 \end{aligned}$$

The answer is $\boxed{5 \log_2 3 \text{ years}}$.

8. Sketch each angle in standard position on the unit circle.

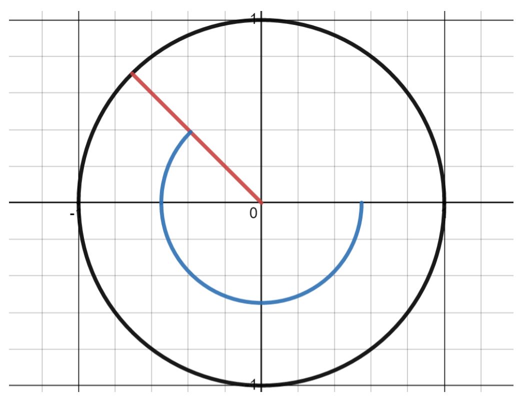
(a) $\theta = \frac{4\pi}{3}$ (2 pts)

Solution:



(b) $\theta = -\frac{5\pi}{4}$ (2 pts)

Solution:



9. The following are unrelated.

(a) For an angle θ in standard position, suppose we know $\cos(\theta) > 0$ and $\sin(\theta) < 0$. What quadrant does the terminal side of θ lie? (3 pts)

Solution:

Since $\cos(\theta) > 0$ is only true in quadrants I and IV, and $\sin(\theta) < 0$ is only true in quadrants III and IV, θ must be in $\boxed{\text{Quadrant IV.}}$

(b) Find an angle between 0 and 2π that is co-terminal with $\theta = \frac{7\pi}{2}$ (3 pts)

Solution:

Note that $\frac{7\pi}{2} > 2\pi$, so we must subtract multiples of 2π until we obtain a value between 0 and 2π. $\frac{7\pi}{2} - 2\pi = \frac{7\pi}{2} - \frac{4\pi}{2} = \frac{3\pi}{2}$, which is between 0 and 2π. Thus, $\boxed{\frac{3\pi}{2}}$ is an angle between 0 and 2π that is co-terminal with $\frac{7\pi}{2}$.

(c) Find $\cos t$ in terms of $\sin t$ if the terminal point determined by t is in quadrant II. (3 pts)

Solution:

Using the Pythagorean identity relating $\cos t$ and $\sin t$:

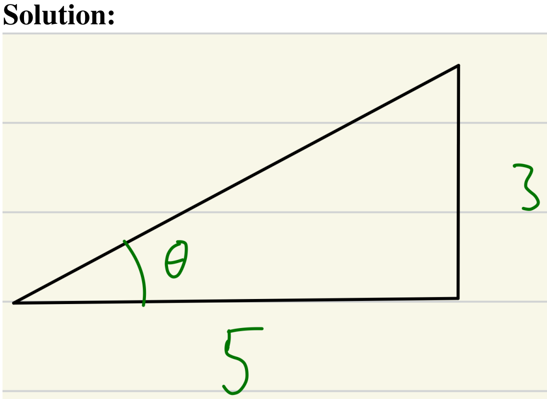
$$\begin{aligned} \sin^2 t + \cos^2 t &= 1 \\ \cos^2 t &= 1 - \sin^2 t \\ \cos t &= \pm\sqrt{1 - \sin^2 t}. \end{aligned}$$

Since $\cos t < 0$ in quadrant II, we take the negative answer: $\boxed{\cos t = -\sqrt{1 - \sin^2 t}}$

10. Answer the following for $\tan(\theta) = \frac{3}{5}$.

(a) Sketch a triangle that has acute angle θ (3 pts).

Solution:



(b) Find $\cos\theta$ (3 pts)

Solution:

Since the triangle is a right triangle it satisfies the Pythagorean theorem. Labeling the hypotenuse h, we get:

$$\begin{aligned} h^2 &= 3^2 + 5^2 \\ h^2 &= 34 \\ h &= \sqrt{34} \end{aligned}$$

So $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \boxed{\frac{5}{\sqrt{34}}}$.

(c) Find $\sec\theta$ (3 pts)

Solution:

$$\sec\theta = \frac{1}{\cos(\theta)} = \frac{1}{\frac{5}{\sqrt{34}}} = \boxed{\frac{\sqrt{34}}{5}}.$$

11. Find the exact value of each of the following. If a value does not exist write DNE.

(a) $\sin(180^\circ)$ (3 pts)

Solution:

$$\sin(180^\circ) = \boxed{0}$$

(b) $\cos\left(\frac{2\pi}{3}\right)$ (3 pts)

Solution:

$$\cos\left(\frac{2\pi}{3}\right) = \boxed{-\frac{1}{2}}$$

(c) $\tan\left(-\frac{5\pi}{6}\right)$ (3 pts)

Solution:

$$\tan\left(-\frac{5\pi}{6}\right) = \boxed{\frac{1}{\sqrt{3}}}$$

(d) $\csc(240^\circ)$ (3 pts)

Solution:

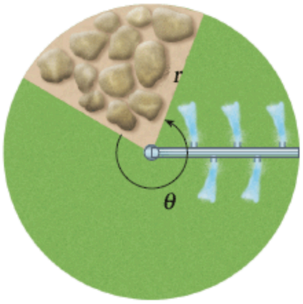
$$\csc(240^\circ) = \boxed{-\frac{2}{\sqrt{3}}}$$

(e) $\cos\left(\frac{7\pi}{4}\right)$ (3 pts)

Solution:

$$\cos\left(\frac{7\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$$

12. An irrigation system uses a straight sprinkler pipe $r = 6$ ft long that pivots around a central point as shown. Because of an obstacle the pipe is allowed to pivot through $\theta = 260°$ only. Find the area irrigated by this system. (4 pts)



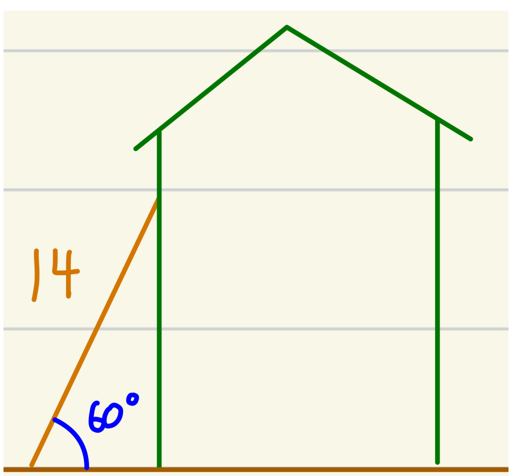
Solution:

We will utilize the area of a circular arc formula $A = \frac{1}{2}r^2\theta$ where θ is in radians. We start by converting the angle into radians: $260° \cdot \frac{\pi}{180°} = \frac{13\pi}{9}$. So the area is given by $A = \frac{1}{2}\left(6^2\right)\frac{13\pi}{9} = \boxed{26\pi}$ ft^2.

13. A 14-ft ladder leans against a building so that the angle between the ground and the ladder is 60°. How high does the ladder reach on the building? (4 pts)

Solution:

We start by sketching a picture:



Let y be the height the ladder reaches on the side of the building. We see a right triangle is formed and can write out the relationship: $\sin(60°) = \frac{y}{14}$. So $y = 14\sin(60°) = 14\left(\frac{\sqrt{3}}{2}\right) = \boxed{7\sqrt{2}}$ ft.