INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **justify all answers**. A correct answer with incorrect work or no justification may receive no credit. Books, notes, and electronic devices are not permitted while taking the exam. The exam is worth 100 points.

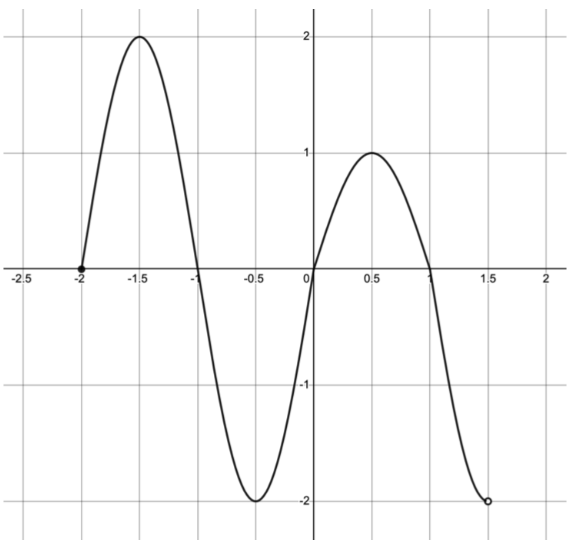
Potentially useful formulas:

(i) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

(ii) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

(iii) Equation of a circle: $(x-h)^2 + (y-k)^2 = r^2$

1. Answer the following for the given graph of a function f (no justification is needed for this problem) (11pts):



(a) Identify the domain of f.
Solution:

$$\boxed{D : [-2, 1.5)}$$

(b) Identify the the range of f.
Solution:

$$\boxed{R : [-2, 2]}$$

(c) Find $f(-2)$ if it exists.
Solution:

$$\boxed{f(-2) = 0}$$

(d) Find $(f+f)(-1.5)$
Solution:

$$(f+f)(-1.5) = f(-1.5) + f(-1.5) = 2 + 2 = \boxed{4}$$

(e) Is f odd, even, or neither?

Solution:

$\boxed{\text{Neither}}$ as it is not symmetric about the origin nor y-axis

(f) Find $(f \circ f)(0.5)$.

Solution:

$(f \circ f)(0.5) = f(f(0.5)) = f(1) = \boxed{0}$

(g) f is not one-to-one. Why isn't it?

Solution:

$\boxed{\text{It does not pass the horizontal line test. For example, f(0) = f(1) = 0 but the } x \text{ values } \mathbf{0 \neq 1}}$

(h) Identify a restriction of the domain so that f is one-to-one and has the same range as in part (b).

Solution:

Let the restriction be $\boxed{D : [-1.5, -0.5]}$

(i) Use your domain restriction to calculate $f^{-1}(0)$.

Solution:

$\boxed{f^{-1}(0) = -1}$. This is because the domain and range flip when dealing with the inverse. So, an input of $x = 0$ into the inverse means it is an output value for the original function f. From the above domain restriction, f is zero at $x = -1$

(j) **True** or **False**: The only local minimum of the function f is $y = 0$.

Solution:

$\boxed{\text{False, } y = -2 \text{ is also a minimum}}$

(k) Find the x-values where $f(x) > 0$. Give your answer in interval notation.

Solution:

$\boxed{(-2, -1) \cup (0, 1)}$

2. The following are unrelated: (10 pts)

(a) Find the equation of the line through the points $(-2, 3)$ and $(1, -4)$.
Solution:

First we need to find the slope between the two points:

$$\begin{aligned} m &= \frac{-4-3}{1-(-2)} \\ &= \frac{-7}{1+2} \\ &= -\frac{7}{3} \end{aligned}$$

Next, we can use either one of the points to put into the point-slope form (i.e. $y - y_1 = m(x - x_1)$)

$$\begin{aligned} y-3 &= -\frac{7}{3}(x-(-2)) \\ y-3 &= -\frac{7}{3}(x+2) \\ y-3 &= -\frac{7}{3}x - \frac{14}{3} \\ y &= -\frac{7}{3}x - \frac{14}{3} + 3 \\ y &= -\frac{7}{3}x - \frac{14}{3} + \frac{9}{3} \\ y &= -\frac{7}{3}x - \frac{5}{3} \end{aligned}$$

Thus $\boxed{y = -\frac{7}{3}x - \frac{5}{3}}$

(b) Find the equation of the line parallel to the x-axis and through the point $(-2, -4)$.
Solution:

Because the line is parallel to the x-axis, that means the slope is $m = 0$. So,

$$\begin{aligned} y-(-4) &= 0 * (x-(-2)) \\ y+4 &= 0 \\ y &= -4 \end{aligned}$$

Thus $\boxed{y = -4}$

(c) Find all value(s) of b such that the distance between the two points, $(2, 0)$ and $(3, b)$, is 2.
Solution:

$$\begin{aligned} 2 &= \sqrt{(3-2)^2 + (b-0)^2} \\ 2 &= \sqrt{(1)^2 + (b)^2} \\ 2 &= \sqrt{1+b^2} \\ 2^2 &= 1+b^2 \\ 4 &= 1+b^2 \\ 4-1 &= b^2 \\ 3 &= b^2 \\ \pm\sqrt{3} &= b \end{aligned}$$

Thus $\boxed{b = \pm\sqrt{3}}$

3. Find the center and radius for: $x^2 + y^2 - 4y = 3$. (3 pts)

Solution:

Need to complete the square

$$\begin{aligned} x^2 + y^2 - 4y &= 3 \\ x^2 + y^2 - 4y + 4 &= 3 + 4 \\ x^2 + (y-2)^2 &= 7 \end{aligned}$$

Thus $\boxed{C : (0,2) \quad r = \sqrt{7}}$

4. Find the domain of the following functions. Express your answers in interval notation. (15 pts)

(a) $n(x) = 4x + |3x| + 3$

Solution:

$n(t)$ has a domain of all real numbers since there is no possibility of division by zero or taking the even root of a negative number. The domain is $\boxed{(-\infty, \infty)}$.

(b) $h(x) = \dfrac{x\sqrt{1-x}}{x+2}$

Solution:

The domain is all real numbers except when the denominator is 0 and the expression under the square root is negative. That is, we need $x + 2 \neq 0$ and $1 - x \geq 0$. Solving the first equation yields $x \neq -2$, and solving the second equation yields $1 \geq x$. The domain is then $\boxed{(-\infty, -2) \cup (-2, 1]}$.

(c) $s(x) = \dfrac{x^2 - 4x + 3}{x^2 - 3x}$

Solution:

The domain is all real numbers except when $x^2 - 3x = 0$. We solve the equation:

$$\begin{aligned} x^2 - 3x &= 0 \\ x(x-3) &= 0 \end{aligned}$$

This results in a domain of all real numbers but we must exclude $x = 0$ and $x = 3$. In interval notation: $\boxed{(-\infty, 0) \cup (0, 3) \cup (3, \infty)}$.

5. For $f(x) = \dfrac{1}{x+2}$ and $g(x) = x^2 - 4$, find the following: (10 pts)

   (a) Find $g(2)$
   Solution:

   $g(2) = 2^2 - 4 = 4 - 4 = \boxed{0}$.

   (b) Find $g(a)$
   Solution:

   $g(a) = \boxed{a^2 - 4}$.

   (c) Find $(f \circ g)(x)$.
   Solution:

   $(f \circ g)(x) = f(g(x)) = f(x^2 - 4) = \dfrac{1}{(x^2-4)+2} = \boxed{\dfrac{1}{x^2-2}}$.

   (d) $(fg)(x)$ and its domain.
   Solution:

   $(fg)(x) = f(x)g(x) = \dfrac{1}{x+2} \cdot x^2 - 4 = \dfrac{(1)}{x+2} \cdot (x-2)(x+2) = \boxed{x-2}$
   Since -2 is not in the domain of $f(x)$, it can't be in the domain of $(fg)(x)$, so the domain of $(fg)(x)$ is $\boxed{(-\infty, -2) \cup (-2, \infty)}$.

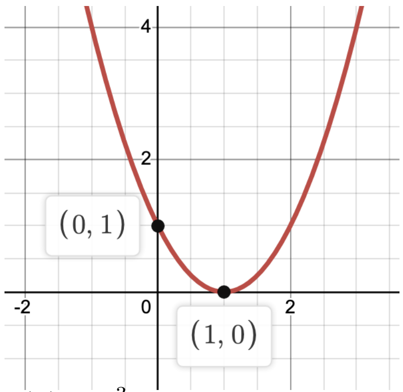
6. Find the average rate of change of the function $g(x) = x^2 - 3$ between $x = a$ and $x = a + h$. (4 pts)
   Solution:

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a+h) - g(a)}{(a+h) - a} \\
&= \frac{((a+h)^2 - 3) - (a^2 - 3)}{h} \\
&= \frac{a^2 + 2ah + h^2 - 3 - (a^2 - 3)}{h} \\
&= \frac{a^2 + 2ah + h^2 - 3 - a^2 + 3}{h} \\
&= \frac{2ah + h^2}{h} \\
&= \frac{h(2a+h)}{h} \\
&= \boxed{2a + h}
\end{aligned}$$

7. Sketch the shape of the graph of each of the following on the provided axes. Make sure to label relevant value(s) on your axe(s) (19 pts)

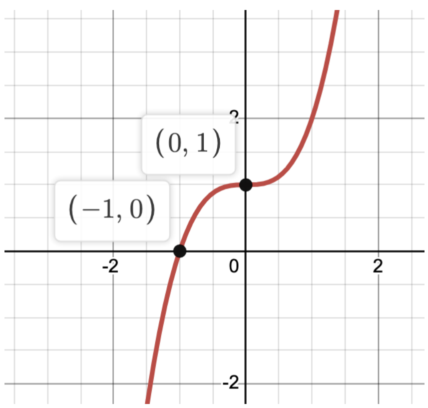
(a) $f(x) = (x-1)^2$

Solution:



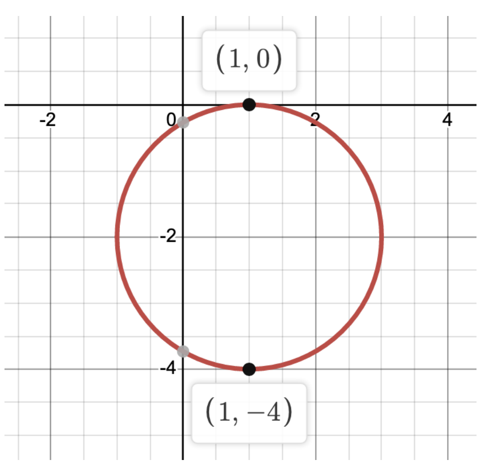
(b) $k(x) = x^3 + 1$

Solution:

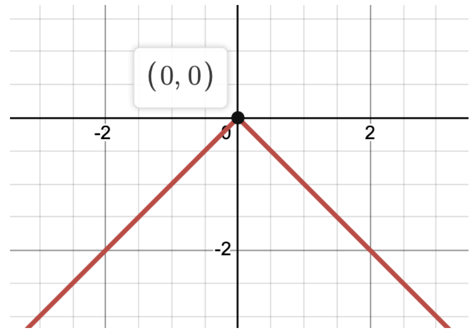


(c) $(x-1)^2 + (y+2)^2 = 4$

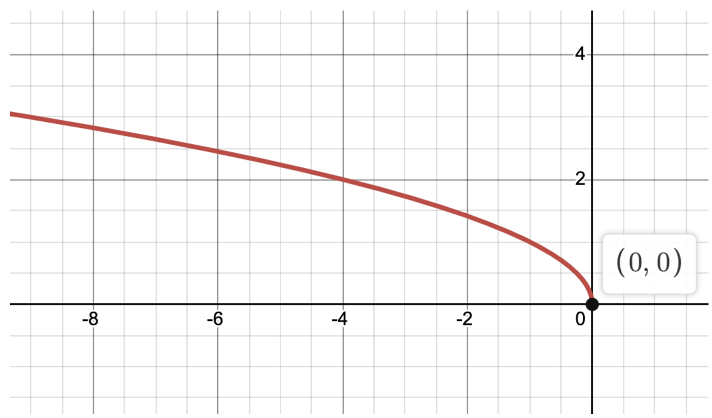
Solution:



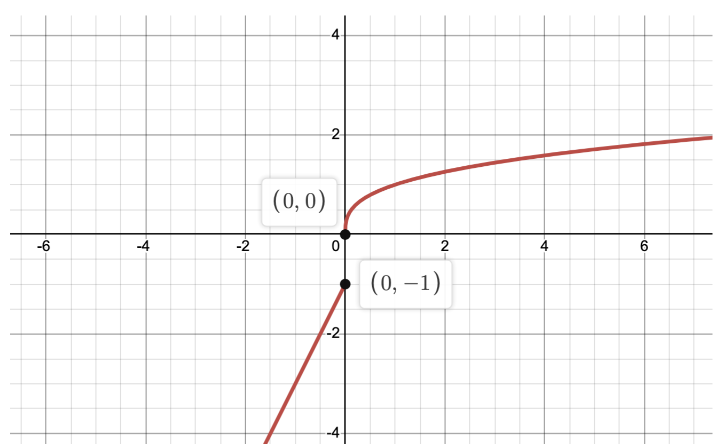
(d) $g(x) = -|x|$



(e) $m(x) = \sqrt{-x}$



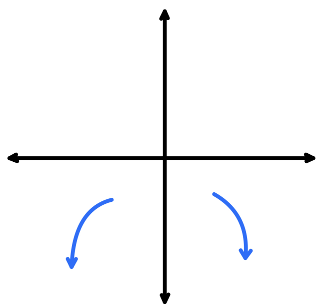
(f) $q(x) = \begin{cases} 2x-1 & \text{if } x \leq 0 \\ \sqrt[3]{x} & \text{if } x > 0 \end{cases}$



8. For $P(x) = -x^4 + 6x^3 - 5x^2$ answer the following. (7 pts)

(a) Indicate on a graph or use arrow notation to indicate the end behavior of $P(x)$.
Solution:

As $x \to \infty$, $f(x) \to -\infty$ As $x \to -\infty$, $f(x) \to -\infty$



(b) Find all zeros and identify the multiplicity of each zero.
Solution:

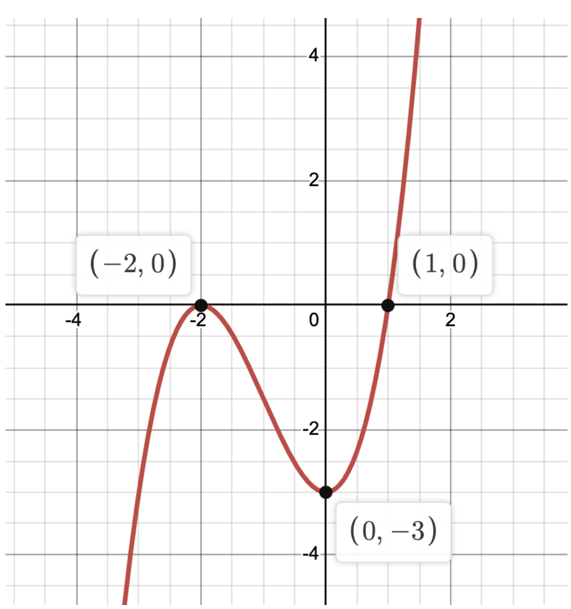
Finding zeros:

$$
\begin{aligned}
0 &= -x^4 + 6x^3 - 5x^2 \\
0 &= -x^2(x^2 - 6x + 5) \\
0 &= -x^2(x - 5)(x - 1)
\end{aligned}
$$

Zeros:
$x = 0$ has a multiplicity of 2
$x = 1$ has a multiplicity of 1
$x = 5$ has a multiplicity of 1

9. Sketch the shape of the graph of a polynomial function, $f(x)$, that satisfies **all** of the information. **Label** all intercepts on the graph. (5 pts)

i. The graph has y-intercept $(0, -3)$.
ii. The graph has end behavior consistent with $y = x^3$.
iii. The graph bounces (touches but does not cross) at $(-2, 0)$ and crosses at $(1, 0)$.
iv. The graph has no other x-intercepts.



10. Use long division to find the quotient and remainder when $2x^4+4x^3-2x^2+2$ is divided by x^2+2x+1. (4 pts)

Solution:

$$
\begin{array}{rl}
 & 2x^2 - \quad 4 \\
\cline{2-2}
x^2+2x+1 \big) & 2x^4+4x^3- \quad 2x^2+0x+\ 2 \\
 & -(\underline{2x^4+4x^3+\ \ 2x^2)} \\
 & \qquad\qquad\qquad -4x^2+0x+\ 2 \\
 & \qquad\qquad\quad -\underline{(-4x^2-8x-4)} \\
 & \qquad\qquad\qquad\qquad\quad 8x+\ 6
\end{array}
$$

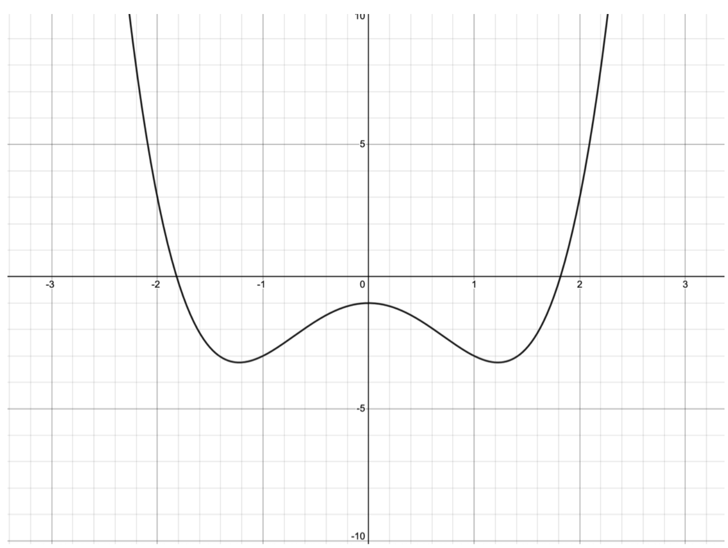
So the quotient is $\boxed{2x^2-4}$ and the remainder is $\boxed{8x+6}$.

11. (a) Is $f(x)=x^4+2x^2-1$ odd, even, or neither? Justify your answer for credit. (4 pts)

Solution:

$f(-x)=(-x)^4+2(-x)^2-1=x^4+2x^2-1=f(x)$ so the function is $\boxed{even}$.

(b) Given the graph of a function below, is the function odd, even, or neither? **No justification is needed.** (3 pts)



Solution:

$\boxed{Even}$ since the graph is symmetric about the y-axis.

12. A wire 10 cm long is cut into two pieces, one of length x and the other of length y. Each piece is bent into the shape of a square. Express the total area of the two squares as a function of x. (5 pts)

Solution:

Since the first square has perimeter x then each side of the square has length $\frac{x}{4}$. The second square has perimeter y so each side of the square has length $\frac{y}{4}$. Thus the area of both squares, A, is $\left(\frac{x}{4}\right)\left(\frac{x}{4}\right)+\left(\frac{y}{4}\right)\left(\frac{y}{4}\right)$. We also know that $10 = x + y$ so $y = 10 - x$. So: $A = \frac{x^2}{16} + \frac{y^2}{16} = \frac{x^2}{16} + \frac{(10-x)^2}{16}$ and the answer is $\boxed{A(x) = \frac{x^2}{16} + \frac{(10-x)^2}{16}}$.

END OF EXAM