INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, electronic devices (such as calculator or other unauthorized electronic resources) are not permitted. **Give all answers in exact form.**

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1. Simplify each of the following. Leave answers without negative exponents. (18 pts)

   (a) $(2a-1)(a-1)-a(a+2)$

   **Solution**

$$\begin{aligned}
&(2a-1)(a-1)-a(a+2) && (1)\\
&= 2a^2-3a+1-a^2-2a && (2)\\
&= a^2-5a+1 && (3)
\end{aligned}$$

   (b) $\left(\dfrac{9q^2p^{-3}}{3^{-1}q^{-4}p^0}\right)(qp)^{-1}$

   **Solution**

$$\begin{aligned}
&\left(\frac{9q^2p^{-3}}{3^{-1}q^{-4}p^0}\right)(qp)^{-1} && (4)\\
&= \left(\frac{9(3)q^2q^4}{p^3}\right)\frac{1}{qp} && (5)\\
&= \left(\frac{27q^6}{p^3}\right)\frac{1}{qp} && (6)\\
&= \frac{27q^5}{p^4} && (7)
\end{aligned}$$

(c) $\sqrt{\sqrt[3]{(x^3y^6)^4}}$

**Solution**

$$\begin{aligned}
&\sqrt{\sqrt[3]{(x^3y^6)^4}} && (8)\\
&= \sqrt{\sqrt[3]{x^{12}y^{24}}} && (9)\\
&= \sqrt{x^4y^8} && (10)\\
&= x^2y^4 && (11)
\end{aligned}$$

(d) $\left(x^{1/2} - y^{1/3}\right)\left(x^{1/2} + y^{1/3}\right)$

**Solution**

$$\begin{aligned}
&\left(x^{1/2} - y^{1/3}\right)\left(x^{1/2} + y^{1/3}\right) && (12)\\
&= x^{1/2}x^{1/2} + x^{1/2}y^{1/3} - y^{1/3}x^{1/2} - y^{1/3}y^{1/3} && (13)\\
&= x - y^{2/3} && (14)
\end{aligned}$$

(e) $\dfrac{x + \frac{1}{x^2}}{1 + \frac{1}{x^3}}$

**Solution**

$$\begin{aligned}
&\frac{x + \frac{1}{x^2}}{1 + \frac{1}{x^3}} && (15)\\
&= \frac{x + \frac{1}{x^2}}{1 + \frac{1}{x^3}} \cdot \frac{x^3}{x^3} && (16)\\
&= \frac{x^4 + x}{x^3 + 1} && (17)\\
&= \frac{x(x^3 + 1)}{x^3 + 1} && (18)\\
&= x && (19)
\end{aligned}$$

(f) $(e^x - y)^2 - ye^x - \ln\left(ye^{y^2}\right)$

**Solution**

$$\begin{aligned}
(e^x - y)^2 - ye^x - \ln\left(ye^{y^2}\right) \quad & (20)\\
= e^{2x} - 2ye^x + y^2 - ye^x - \ln y - \ln\left(e^{y^2}\right) \quad & (21)\\
= e^{2x} - 3ye^x + y^2 - \ln y - y^2 \quad & (22)\\
= e^{2x} - 3ye^x - \ln y \quad & (23)
\end{aligned}$$

2. Solve the following equations for $x$: (25 pts)

(a) $(x-2)(x+3) = 1$

**Solution**

$$\begin{aligned}
(x-2)(x+3) &= 1 && (24)\\
x^2 + x - 6 &= 1 && (25)\\
x^2 + x - 7 &= 0 && (26)\\
x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-7)}}{2(1)} && (27)\\
x &= \frac{-1 \pm \sqrt{29}}{2} && (28)
\end{aligned}$$

(b) $\sqrt{-3x+6} + x = -4$

**Solution**

$$\begin{aligned}
\sqrt{-3x+6} + x &= -4 && (29)\\
\sqrt{-3x+6} &= -4 - x && (30)\\
-3x + 6 &= (-4 - x)^2 && (31)\\
-3x + 6 &= 16 + 8x + x^2 && (32)\\
0 &= x^2 + 11x + 10 && (33)\\
0 &= (x+10)(x+1) && (34)
\end{aligned}$$

Leading to two possible answers: $x = -1, -10$. Checking both answers in the original equation we see that only $x = -10$ solves the original equation.

(c) $\dfrac{1}{2x-1} - \dfrac{\sqrt{2}}{2x+1} = -\dfrac{3}{4x^2-1}$

**Solution**

We start by clearing out the fractions by multiplying both sides of the equation by the LCD: $(2x-1)(2x+1)$.

$$\begin{aligned}
\frac{1}{2x-1} - \frac{\sqrt{2}}{2x+1} &= -\frac{3}{4x^2-1} && (35)\\
2x+1-\sqrt{2}\,(2x-1) &= -3 && (36)\\
2x+1-2\sqrt{2}x+\sqrt{2} &= -3 && (37)\\
2x-2\sqrt{2}x &= -4-\sqrt{2} && (38)\\
x\left(2-2\sqrt{2}\right) &= -4-\sqrt{2} && (39)\\
x &= \frac{-4-\sqrt{2}}{2-2\sqrt{2}} && (40)
\end{aligned}$$

(d) $3^{3x-7} = 9^{x-1}$

**Solution**

$$\begin{aligned}
3^{3x-7} &= 9^{x-1} && (41)\\
3^{3x-7} &= \left(3^2\right)^{x-1} && (42)\\
3^{3x-7} &= 3^{2x-2} && (43)\\
3x-7 &= 2x-2 && (44)\\
x &= 5 && (45)
\end{aligned}$$

(e) $-2\log(2) = \log(3x+1)$

**Solution**

$$\begin{aligned}
-2\log(2) &= \log(3x+1) && (46)\\
\log\left(2^{-2}\right) &= \log(3x+1) && (47)\\
2^{-2} &= 3x+1 && (48)\\
\frac{1}{4} &= 3x+1 && (49)\\
1 &= 12x+4 && (50)\\
-3 &= 12x && (51)\\
-\frac{1}{4} &= x && (52)
\end{aligned}$$

Checking this answer in the original equation we see that it does solve the original equation.

3. For $g(x) = \dfrac{3x-1}{x^6 - 9x^4}$ answer the following (12 pts):

(a) Find the domain of $g(x)$. Give your answer in interval notation.

**Solution**

The domain is all real numbers except where $x^6 - 9x^4 = 0$.

$$x^6 - 9x^4 = 0 \tag{53}$$
$$x^4\left(x^2 - 9\right) = 0 \tag{54}$$
$$x^4\left(x-3\right)\left(x+3\right) = 0 \tag{55}$$

So the domain is all real numbers except $x = -3, 0, 3$. In interval notation: $(-\infty, -3) \cup (-3, 0) \cup (0, 3) \cup (3, \infty)$.

(b) Find the $x$, $y$-coordinates for any hole(s). If there are none write NONE.

**Solution**

Factoring the numerator and denominator: $g(x) = \dfrac{3x-1}{x^4\left(x-3\right)\left(x+3\right)}$. Since no terms cancel there are no holes. NONE.

(c) Find any horizontal or slant asymptotes. If there are none write NONE.

**Solution**

Looking at the leading terms in numerator and denominator: $g(x) = \dfrac{3x-1}{x^6 - 9x^4} \approx \dfrac{3x}{x^6} = \dfrac{3}{x^5}$ as $x \to \infty$. So $g(x) \to 0$ as $x \to \infty$. Thus there is a horizontal asymptote of $y = 0$.

(d) Find any vertical asymptotes. If there are none write NONE.

**Solution**

Since there are no holes then the vertical asymptotes occur when $x^6 - 9x^4 = x^4\left(x-3\right)\left(x+3\right) = 0$. The vertical asymptotes are: $x = 0$, $x = -3$, and $x = 3$.

4. Consider the function $P(x) = -x^4 + x^3 + 6x^2$. Answer the following: (11 pts)

   (a) Find all $x$ and $y$-intercepts.

   **Solution**

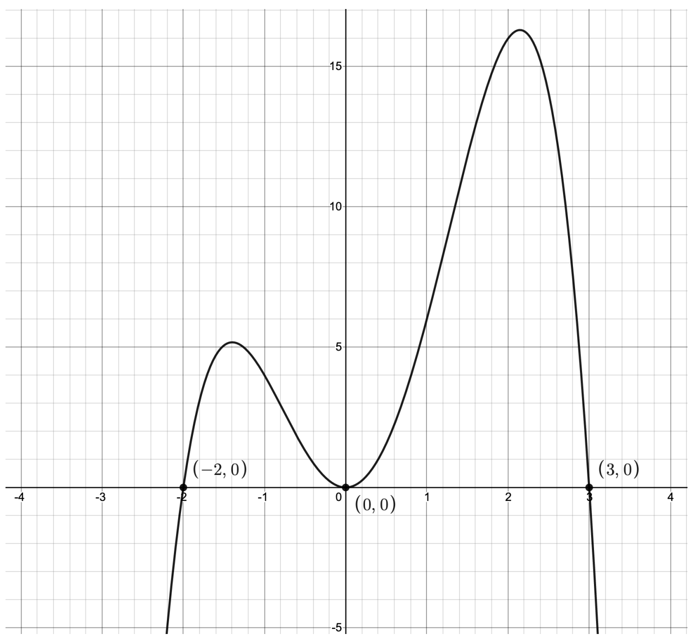
   The $y$-intercept occurs when $x = 0$. $P(0) = -\left(0^4\right) + 0^3 + 6\left(0^2\right) = 0$. The $y$-intercept is $(0, 0)$.

   The $x$-intercept(s) occurs when $y = 0$. $P(x) = 0 = -x^4 + x^3 + 6x^2 = -x^2\left(x^2 - x - 6\right) = -x^2(x-3)(x+2)$. So the $x$-intercepts occur at $(0, 0)$, $(3, 0)$, and $(-2, 0)$.

   (b) Identify the end behavior (either using arrow notation or depicting on a graph).

   The end behavior is controlled by the leading term: $-x^4$. So $P(x) \to -\infty$ as $x \to -\infty$ and $P(x) \to -\infty$ as $x \to \infty$.

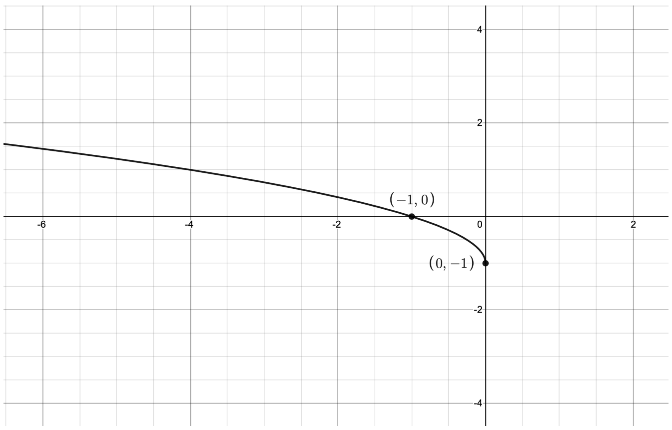
   (c) Sketch the graph of $P(x)$ be sure to label all $x$ and $y$-intercepts.



5. Sketch the graph of the following functions. Label all intercepts and asymptotes as appropriate. (13 pts)

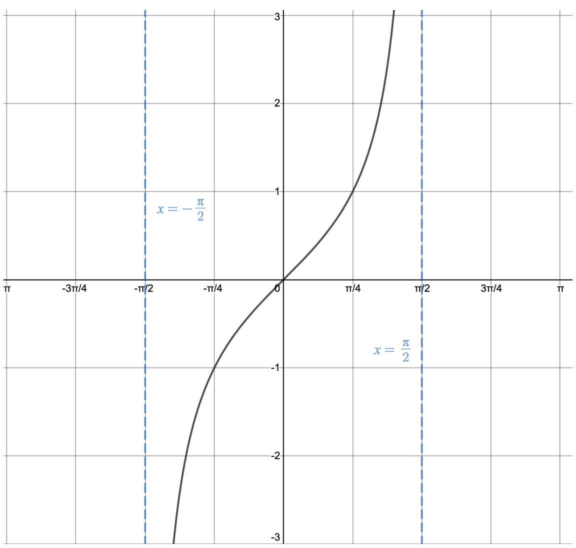
(a) $f(x) = \sqrt{-x} - 1$.

**Solution**



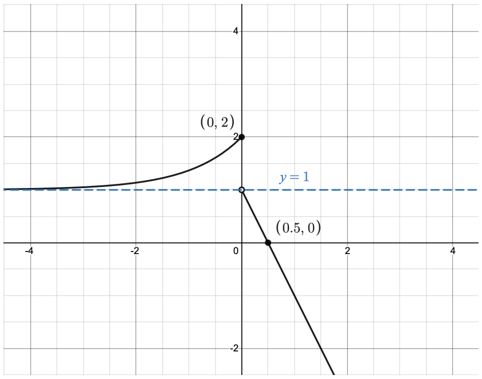

(b) $h(x) = \tan(x)$ on the restricted domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

**Solution**



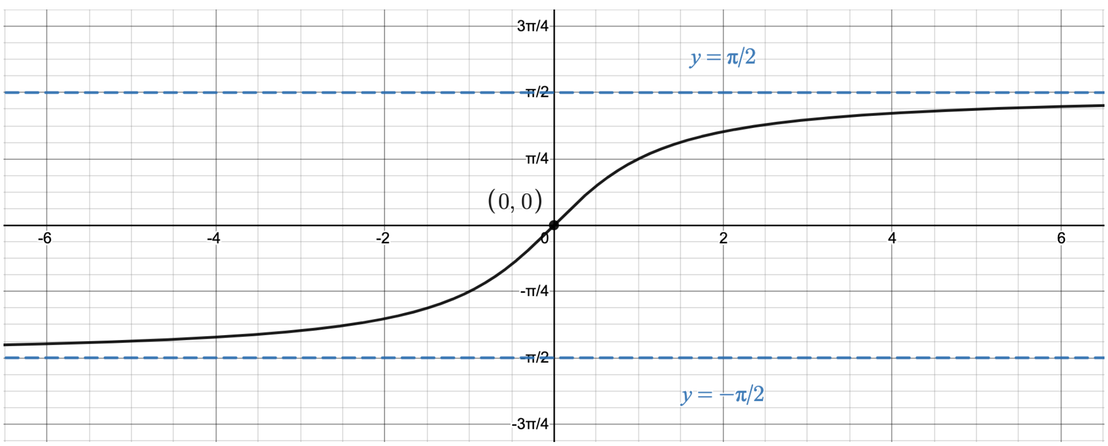
(c) $q(x) = \begin{cases} e^x + 1 & \text{if} \quad x \leq 0 \\ -2x + 1 & \text{if} \quad x > 0 \end{cases}$

**Solution**




(d) $k(x) = \tan^{-1}(x)$

**Solution**




6. Given $f(x) = \sqrt{x}$ and $g(x) = \log(x)$ answer the following. (8 pts)

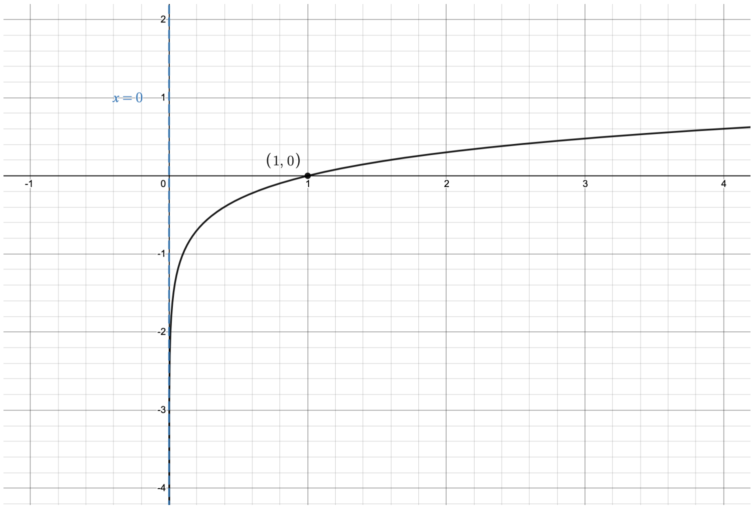
(a) Find $(f + g)(1)$.

**Solution**

$$(f + g)(1) \tag{56}$$
$$= f(1) + g(1) \tag{57}$$
$$= \sqrt{1} + \log(1) \tag{58}$$
$$= 1 + 0 \tag{59}$$
$$= 1 \tag{60}$$

(b) Sketch a graph of $g(x)$. Label all asymptote(s) and intercept(s).

**Solution**




(c) Find $(f \circ g)(x)$ and find the domain.

**Solution**

$(f \circ g)(x) = f(g(x)) = f(\log(x)) = \sqrt{\log(x)}$. The domain of the composition of functions is the intersection of the domains of $g(x)$ and $\sqrt{\log(x)}$. The domain of $g(x)$ is $(0, \infty)$ and the domain of $\sqrt{\log(x)}$ is found when $\log(x) \geq 0$ which, we can see from the picture in part (b) of this problem, occurs when $x \geq 1$. So the domain of $(f \circ g)(x)$ is $[1, \infty)$.

7. Find the exact value: (14 pts)

(a) $\cos\left(\frac{5\pi}{6}\right)$

**Solution**

$$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

(b) $\csc\left(-\frac{4\pi}{3}\right)$

**Solution**

$$\csc\left(-\frac{4\pi}{3}\right) = \frac{2}{\sqrt{3}}$$

(c) $\tan^{-1}\left(\sqrt{3}\right)$

**Solution**

$$\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

(d) $\arccos(1)$

**Solution**

$\arccos(1) = 0$

(e) $\cos\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$

**Solution**

$$\cos\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$$

(f) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$

**Solution**

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

8. For a specific angle $\theta$ suppose we know that $\cos\theta < 0$ and $\theta$ lies in the interval $[0, \pi]$. What quadrant does $\theta$ lie in? (4 pts)

**Solution**

$\cos\theta < 0$ in quadrants II and III and $[0, \pi]$ specifies angles in quadrants I and II so the only quadrant that satisfies both pieces of information is Quadrant II.

9. Verify the identity: $\dfrac{\csc^2\theta}{1+\tan^2\theta} = \cot^2\theta$. (6 pts)

**Solution**

Starting with the left hand side and recognizing that $1 + \tan^2\theta = \sec^2\theta$ we get:

$$\frac{\csc^2\theta}{1+\tan^2\theta} \tag{61}$$

$$= \frac{\csc^2\theta}{\sec^2\theta} \tag{62}$$

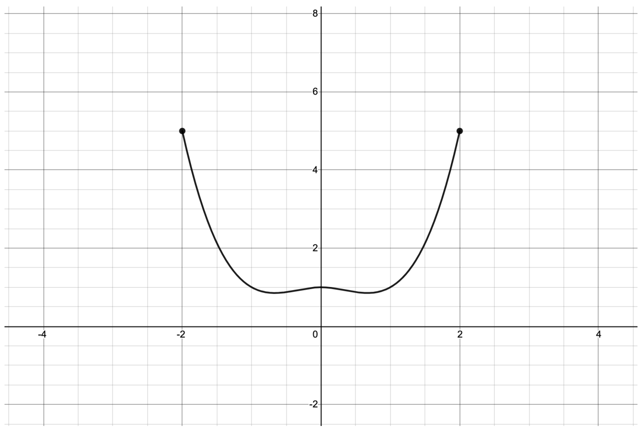
$$= \frac{\frac{1}{\sin^2\theta}}{\frac{1}{\cos^2\theta}} \tag{63}$$

$$= \frac{1}{\sin^2\theta} \cdot \frac{\cos^2\theta}{1} \tag{64}$$

$$= \cot^2\theta_{//} \tag{65}$$

10. The following questions are on the topic of symmetry of a function but are otherwise unrelated. (9 pts)

(a) i. For the graph of $g(x)$ below, with domain $[-2, 2]$, is this the graph of an odd, even, or neither function? No justification is needed.




**Solution**

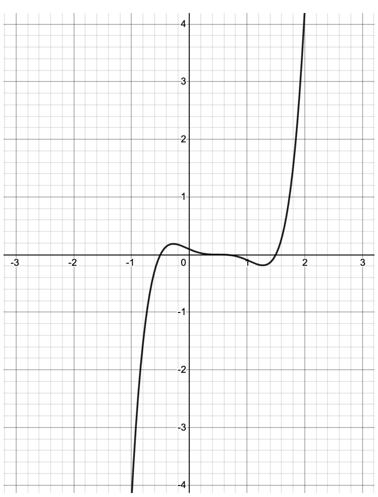
Even

ii. Does the function $g^{-1}(x)$ exist? Give a brief explanation of why or why not.

**Solution**

The function $g^{-1}(x)$ does not exist since $g(x)$ is not one-to-one.

(b) For the graph of $h(x)$ below is this the graph of an odd, even, or neither function? No justification is needed.




**Solution**

Neither

(c) Is $f(x) = \sin x + \cos x$ odd, even, or neither? As usual justify answer for credit.

**Solution**

$f(-x) = \sin(-x) + \cos(-x) = -\sin(x) + \cos(x)$ which is not equal to $f(x)$ or $-f(x)$ so $f(x)$ is neither odd nor even.

11. Find all solutions to the following equations: (10 pts)

(a) $\cos\theta + 2\sin\theta\cos\theta = 0$

**Solution**

$$\cos\theta + 2\sin\theta\cos\theta = 0 \tag{66}$$
$$\cos\theta\,(1 + 2\sin\theta) = 0 \tag{67}$$

Resulting in equations $\cos\theta = 0$ and $1 + 2\sin\theta = 0$ or $\sin\theta = -\frac{1}{2}$. Let $k$ be any integer. These equations result in solutions: $\theta = \frac{\pi}{2} + 2k\pi$, $\theta = \frac{3\pi}{2} + 2k\pi$, $\theta = \frac{7\pi}{6} + 2k\pi$, and $\theta = \frac{11\pi}{6} + 2k\pi$. Which can be condensed into solutions: $\theta = \frac{\pi}{2} + k\pi$, $\theta = \frac{7\pi}{6} + 2k\pi$, and $\theta = \frac{11\pi}{6} + 2k\pi$.

(b) $\cos\left(\frac{\theta}{3}\right) = \frac{1}{2}$

**Solution**

Let $k$ be any integer. $\cos\left(\frac{\theta}{3}\right) = \frac{1}{2}$ when $\frac{\theta}{3} = \frac{\pi}{3} + 2k\pi$ and $\frac{\theta}{3} = \frac{5\pi}{3} + 2k\pi$. Solving for $\theta$ and we get $\theta = \pi + 6k\pi$ and $\theta = 5\pi + 6k\pi$.

12. Find the exact value for each: (10 pts)

(a) $\cos^2(22.5^\circ) - \sin^2(22.5^\circ)$

**Solution**

By a double angle formula: $\cos^2(22.5^\circ) - \sin^2(22.5^\circ) = \cos(2 \cdot 22.5^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}$.

(b) $\sin\left(-\frac{\pi}{8}\right)$

**Solution**

Using a half angle formula and noting $\sin\left(-\frac{\pi}{8}\right) < 0$ we get $\sin\left(\frac{-\frac{\pi}{4}}{2}\right) = -\sqrt{\frac{1-\cos\left(-\frac{\pi}{4}\right)}{2}} = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$.

13. For $f(x) = 3\sin\left(x - \frac{\pi}{3}\right)$ (10 pts)

(a) Identify the amplitude.

**Solution**

3

(b) Identify the period.

**Solution**

$2\pi$

(c) Identify the phase shift.

**Solution**

$\frac{\pi}{3}$

(d) Sketch one cycle of the graph of $f(x)$. Label at least five $x$-values on the $x$-axis and amplitude values on the $y$-axis to receive full credit.