1. Simplify each of the following. Leave answers without negative exponents. (18 pts)

(a) \((2a - 1)(a - 1) - a(a + 2)\)

\[
(2a - 1)(a - 1) - a(a + 2) = 2a^2 - 3a + 1 - a^2 - 2a
\]

\[
= a^2 - 5a + 1
\]

(b) \(\left( \frac{9q^2p^{-3}}{3^{-1}q^{-4}p^0} \right)(qp)^{-1}\)

\[
\left( \frac{9q^2p^{-3}}{3^{-1}q^{-4}p^0} \right)(qp)^{-1} = \left( \frac{9(3)q^2q^4}{p^3} \right) \frac{1}{qp}
\]

\[
= \left( \frac{27q^6}{p^4} \right) \frac{1}{qp}
\]

\[
= \frac{27q^5}{p^4}
\]
(c) $\sqrt[3]{(x^3 y^6)^4}$

Solution

\[
\sqrt[3]{(x^3 y^6)^4} = \sqrt[3]{x^{12} y^{24}} = x^4 y^8 = x^2 y^4
\]

(d) $\left( x^{1/2} - y^{1/3} \right) \left( x^{1/2} + y^{1/3} \right)$

Solution

\[
\left( x^{1/2} - y^{1/3} \right) \left( x^{1/2} + y^{1/3} \right) = x^{1/2} x^{1/2} + x^{1/2} y^{1/3} - y^{1/3} x^{1/2} - y^{1/3} y^{1/3} = x - y^{2/3}
\]

(e) $\frac{x + \frac{1}{x^2}}{1 + \frac{1}{x^7}}$

Solution

\[
\frac{x + \frac{1}{x^2}}{1 + \frac{1}{x^7}} = \frac{x + \frac{1}{x^2} \cdot x^3}{1 + \frac{1}{x^7} \cdot x^3} = \frac{x^4 + x}{x^3 + 1} = \frac{x(x^3 + 1)}{x^3 + 1} = x
\]
(f) \( (e^x - y)^2 - ye^x - \ln \left( ye^{y^2} \right) \)

Solution

\[
(e^x - y)^2 - ye^x - \ln \left( ye^{y^2} \right) \\
= e^{2x} - 2ye^x + y^2 - ye^x - \ln y - \ln \left( ye^{y^2} \right) \\
= e^{2x} - 3ye^x + y^2 - \ln y - y^2 \\
= e^{2x} - 3ye^x - \ln y
\]  

(20) 

(21) 

(22) 

(23)

2. Solve the following equations for \( x \): (25 pts)

(a) \( (x - 2)(x + 3) = 1 \)

Solution

\[
(x - 2)(x + 3) = 1 \\
x^2 + x - 6 = 1 \\
x^2 + x - 7 = 0
\]  

(24) 

(25) 

(26) 

\[
x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-7)}}{2(1)} \\
x = \frac{-1 \pm \sqrt{29}}{2}
\]  

(27) 

(28)

(b) \( \sqrt{-3x + 6} + x = -4 \)

Solution

\[
\sqrt{-3x + 6} + x = -4 \\
\sqrt{-3x + 6} = -4 - x \\
-3x + 6 = (-4 - x)^2 \\
-3x + 6 = 16 + 8x + x^2 \\
0 = x^2 + 11x + 10 \\
0 = (x + 10)(x + 1)
\]  

(29) 

(30) 

(31) 

(32) 

(33) 

(34)

Leading to two possible answers: \( x = -1, -10 \). Checking both answers in the original equation we see that only \( x = -10 \) solves the original equation.
(c) \( \frac{1}{2x - 1} - \frac{\sqrt{2}}{2x + 1} = -\frac{3}{4x^2 - 1} \)

Solution

We start by clearing out the fractions by multiplying both sides of the equation by the LCD: \((2x - 1)(2x + 1)\).

\[
\begin{align*}
\frac{1}{2x - 1} - \frac{\sqrt{2}}{2x + 1} &= -\frac{3}{4x^2 - 1} \\
2x + 1 - \sqrt{2}(2x - 1) &= -3 \\
2x + 1 - 2\sqrt{2}x + \sqrt{2} &= -3 \\
2x - 2\sqrt{2}x &= -4 - \sqrt{2} \\
x(2 - 2\sqrt{2}) &= -4 - \sqrt{2} \\
x &= \frac{-4 - \sqrt{2}}{2 - 2\sqrt{2}}
\end{align*}
\]

(d) \(3^{3x - 7} = 9^{x - 1}\)

Solution

\[
\begin{align*}
3^{3x - 7} &= 9^{x - 1} \\
3^{3x - 7} &= (3^2)^{x - 1} \\
3^{3x - 7} &= 3^{2x - 2} \\
3x - 7 &= 2x - 2 \\
x &= 5
\end{align*}
\]

(e) \(-2 \log(2) = \log(3x + 1)\)

Solution

\[
\begin{align*}
-2 \log(2) &= \log(3x + 1) \\
\log(2^{-2}) &= \log(3x + 1) \\
2^{-2} &= 3x + 1 \\
\frac{1}{4} &= 3x + 1 \\
1 &= 12x + 4 \\
-3 &= 12x \\
-\frac{1}{4} &= x
\end{align*}
\]

Checking this answer in the original equation we see that it does solve the original equation.
3. For \( g(x) = \frac{3x - 1}{x^6 - 9x^4} \) answer the following (12 pts):

(a) Find the domain of \( g(x) \). Give your answer in interval notation.

Solution

The domain is all real numbers except where \( x^6 - 9x^4 = 0 \).

\[
\begin{align*}
x^6 - 9x^4 &= 0 \\
x^4 (x^2 - 9) &= 0 \\
x^4 (x - 3) (x + 3) &= 0
\end{align*}
\]

So the domain is all real numbers except \( x = -3, 0, 3 \). In interval notation: \( (-\infty, -3) \cup (-3, 0) \cup (0, 3) \cup (3, \infty) \).

(b) Find the \( x, y \)-coordinates for any hole(s). If there are none write NONE.

Solution

Factoring the numerator and denominator: \( g(x) = \frac{3x - 1}{x^4 (x - 3) (x + 3)} \). Since no terms cancel there are no holes. NONE.

(c) Find any horizontal or slant asymptotes. If there are none write NONE.

Solution

Looking at the leading terms in numerator and denominator: \( g(x) = \frac{3x - 1}{x^6 - 9x^4} \approx \frac{3x}{x^6} = \frac{3}{x^5} \) as \( x \to \infty \). So \( g(x) \to 0 \) as \( x \to \infty \). Thus there is a horizontal asymptote of \( y = 0 \).

(d) Find any vertical asymptotes. If there are none write NONE.

Solution

Since there are no holes then the vertical asymptotes occur when \( x^6 - 9x^4 = x^4 (x - 3) (x + 3) = 0 \). The vertical asymptotes are: \( x = 0, x = -3, \) and \( x = 3 \).
4. Consider the function \( P(x) = -x^4 + x^3 + 6x^2 \). Answer the following: (11 pts)

(a) Find all \( x \) and \( y \)-intercepts.

**Solution**

The \( y \)-intercept occurs when \( x = 0 \). \( P(0) = -(0^4) + 0^3 + 6(0^2) = 0 \). The \( y \)-intercept is \((0, 0)\).

The \( x \)-intercept(s) occurs when \( y = 0 \). \( P(x) = 0 = -x^4 + x^3 + 6x^2 = -x^2 (x^2 - x - 6) = -x^2(x - 3)(x + 2) \).

So the \( x \)-intercepts occur at \((0, 0)\), \((3, 0)\), and \((-2, 0)\).

(b) Identify the end behavior (either using arrow notation or depicting on a graph).

The end behavior is controlled by the leading term: \(-x^4\). So \( P(x) \to -\infty \) as \( x \to -\infty \) and \( P(x) \to -\infty \) as \( x \to \infty \).

(c) Sketch the graph of \( P(x) \) be sure to label all \( x \) and \( y \)-intercepts.
5. Sketch the graph of the following functions. Label all intercepts and asymptotes as appropriate. (13 pts)

(a) \( f(x) = \sqrt{-x} - 1 \).

Solution

(b) \( h(x) = \tan(x) \) on the restricted domain \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \).

Solution
(c) \( q(x) = \begin{cases} e^x + 1 & \text{if } x \leq 0 \\ -2x + 1 & \text{if } x > 0 \end{cases} \)

**Solution**

\[
\begin{array}{c}
\text{(0, 2)} \\
\text{(0.5, 0)} \\
\end{array}
\]

(d) \( k(x) = \tan^{-1}(x) \)

**Solution**

\[
\begin{array}{c}
\text{(0, 0)} \\
\text{y = \pi/2} \\
\text{y = -\pi/2} \\
\end{array}
\]

6. Given \( f(x) = \sqrt{x} \) and \( g(x) = \log(x) \) answer the following. (8 pts)

(a) Find \((f + g)(1)\).

**Solution**

\[
(f + g)(1) = f(1) + g(1) = \sqrt{1} + \log(1) = 1 + 0 = 1
\]
(b) Sketch a graph of $g(x)$. Label all asymptote(s) and intercept(s).

Solution

(c) Find $(f \circ g)(x)$ and find the domain.

Solution

$$(f \circ g)(x) = f(g(x)) = f(\log(x)) = \sqrt{\log(x)}.$$ The domain of the composition of functions is the intersection of the domains of $g(x)$ and $\sqrt{\log(x)}$. The domain of $g(x)$ is $(0, \infty)$ and the domain of $\sqrt{\log(x)}$ is found when $\log(x) \geq 0$ which, we can see from the picture in part (b) of this problem, occurs when $x \geq 1$. So the domain of $(f \circ g)(x)$ is $[1, \infty)$.

7. Find the exact value: (14 pts)

(a) $\cos \left( \frac{5\pi}{6} \right)$

Solution

$$\cos \left( \frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2}$$

(b) $\csc \left( -\frac{4\pi}{3} \right)$

Solution

$$\csc \left( -\frac{4\pi}{3} \right) = \frac{2}{\sqrt{3}}$$

(c) $\tan^{-1} \left( \sqrt{3} \right)$

Solution

$$\tan^{-1} \left( \sqrt{3} \right) = \frac{\pi}{3}$$
Solution

\[ \arccos(1) = 0 \]

(e) \( \cos \left( \cos^{-1} \left( -\frac{1}{2} \right) \right) \)

Solution

\[ \cos \left( \cos^{-1} \left( -\frac{1}{2} \right) \right) = -\frac{1}{2} \]

(f) \( \sin^{-1} \left( \sin \left( \frac{3\pi}{4} \right) \right) \)

Solution

\[ \sin^{-1} \left( \sin \left( \frac{3\pi}{4} \right) \right) = \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) = \frac{\pi}{4} \]

8. For a specific angle \( \theta \) suppose we know that \( \cos \theta < 0 \) and \( \theta \) lies in the interval \([0, \pi]\). What quadrant does \( \theta \) lie in? (4 pts)

Solution

\( \cos \theta < 0 \) in quadrants II and III and \([0, \pi]\) specifies angles in quadrants I and II so the only quadrant that satisfies both pieces of information is Quadrant II.

9. Verify the identity: \( \frac{\csc^2 \theta}{1 + \tan^2 \theta} = \cot^2 \theta \). (6 pts)

Solution

Starting with the left hand side and recognizing that \( 1 + \tan^2 \theta = \sec^2 \theta \) we get:

\[
\frac{\csc^2 \theta}{1 + \tan^2 \theta} = \frac{\csc^2 \theta}{\sec^2 \theta} = \frac{1}{\sin^2 \theta} \cdot \frac{\cos^2 \theta}{1} = \cot^2 \theta
\]
10. The following questions are on the topic of symmetry of a function but are otherwise unrelated. (9 pts)

(a) i. For the graph of \( g(x) \) below, with domain \([-2, 2]\), is this the graph of an odd, even, or neither function? No justification is needed.

![Graph of g(x)](image)

Solution

Even

ii. Does the function \( g^{-1}(x) \) exist? Give a brief explanation of why or why not.

Solution

The function \( g^{-1}(x) \) does not exist since \( g(x) \) is not one-to-one.

(b) For the graph of \( h(x) \) below is this the graph of an odd, even, or neither function? No justification is needed.

![Graph of h(x)](image)

Solution

Neither
(c) Is \( f(x) = \sin x + \cos x \) odd, even, or neither? As usual justify answer for credit.

Solution

\[ f(-x) = \sin(-x) + \cos(-x) = -\sin(x) + \cos(x) \]
which is not equal to \( f(x) \) or \( -f(x) \) so \( f(x) \) is neither odd nor even.

11. Find all solutions to the following equations: (10 pts)

(a) \( \cos \theta + 2 \sin \theta \cos \theta = 0 \)

Solution

\[
\begin{align*}
\cos \theta + 2 \sin \theta \cos \theta &= 0 \\
\cos \theta (1 + 2 \sin \theta) &= 0
\end{align*}
\]

Resulting in equations \( \cos \theta = 0 \) and \( 1 + 2 \sin \theta = 0 \) or \( \sin \theta = -\frac{1}{2} \). Let \( k \) be any integer. These equations result in solutions: \( \theta = \frac{\pi}{2} + 2k\pi \), \( \theta = \frac{3\pi}{2} + 2k\pi \), \( \theta = \frac{7\pi}{6} + 2k\pi \), and \( \theta = \frac{11\pi}{6} + 2k\pi \). Which can be condensed into solutions: \( \theta = \frac{\pi}{2} + k\pi \), \( \theta = \frac{7\pi}{6} + 2k\pi \), and \( \theta = \frac{11\pi}{6} + 2k\pi \).

(b) \( \cos \left( \frac{\theta}{3} \right) = \frac{1}{2} \)

Solution

Let \( k \) be any integer. \( \cos \left( \frac{\theta}{3} \right) = \frac{1}{2} \) when \( \frac{\theta}{3} = \frac{\pi}{2} + 2k\pi \) and \( \frac{\theta}{3} = \frac{3\pi}{2} + 2k\pi \). Solving for \( \theta \) and we get \( \theta = \pi + 6k\pi \) and \( \theta = 5\pi + 6k\pi \).

12. Find the exact value for each: (10 pts)

(a) \( \cos^2 (22.5^\circ) - \sin^2 (22.5^\circ) \)

Solution

By a double angle formula: \( \cos^2 (22.5^\circ) - \sin^2 (22.5^\circ) = \cos (2 \cdot 22.5^\circ) = \cos (45^\circ) = \frac{\sqrt{2}}{2} \).

(b) \( \sin \left( -\frac{\pi}{8} \right) \)

Solution

Using a half angle formula and noting \( \sin \left( -\frac{\pi}{8} \right) < 0 \) we get \( \sin \left( -\frac{\pi}{8} \right) = -\sqrt{\frac{1 - \cos \left( -\frac{\pi}{4} \right)}{2}} = -\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = -\frac{\sqrt{2} - \sqrt{2}}{2} \).
13. For $f(x) = 3 \sin \left( x - \frac{\pi}{3} \right)$ (10 pts)

(a) Identify the amplitude.

Solution

3

(b) Identify the period.

Solution

$2\pi$

(c) Identify the phase shift.

Solution

$\frac{\pi}{3}$

(d) Sketch one cycle of the graph of $f(x)$. Label at least five $x$-values on the $x$-axis and amplitude values on the $y$-axis to receive full credit.