

INSTRUCTIONS: **Simplify** and **box** all your answers. Write neatly and **show all work**. A correct answer with incorrect or no supporting work may receive no credit. Books, notes, and electronic devices are not permitted while taking the exam. The exam is worth 100 points.

Potentially useful formulas:

(i) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(ii) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(iii) Equation of a circle: $(x - h)^2 + (y - k)^2 = r^2$

DO NOT START UNTIL INSTRUCTED BY A PROCTOR.

WRITE YOUR NAME ON THE NEXT PAGE. JUST BEFORE YOU UPLOAD TO GRADESCOPE WRITE DOWN YOUR UPLOAD TIME ON THE NEXT PAGE.

ONCE A PROCTOR HAS INDICATED YOU CAN START YOU MAY TEAR OFF THIS COVER PAGE AND USE IT AS SCRATCH PAPER.

1. Given the equation of a circle: $(x + 3)^2 + (y - 2)^2 = 15$ answer the following: (8 pts)

(a) Find the coordinates of the center of the circle and state the radius.

Solution:

Comparing the given equation to $(x - h)^2 + (y - k)^2 = r^2$ we get center: $(-3, 2)$. The radius is found if we let $r^2 = 15$ resulting in radius: $r = \sqrt{15}$

(b) Find the equation for the upper half of the circle.

Solution:

$$(x + 3)^2 + (y - 2)^2 = 15 \quad (1)$$

$$(y - 2)^2 = 15 - (x + 3)^2 \quad (2)$$

$$y - 2 = \pm \sqrt{15 - (x + 3)^2} \quad (3)$$

$$y = 2 \pm \sqrt{15 - (x + 3)^2} \quad (4)$$

The equation for the upper half of the circle is given by $y = 2 + \sqrt{15 - (x + 3)^2}$.

2. Find the domain of the following functions. Express your answers in interval notation. (15 pts)

(a) $f(x) = \frac{x^2 - 2x + 1}{x^2 + 2x - 15}$

Solution:

The domain is all real numbers except when $x^2 + 2x - 15 = 0$ We solve the equation:

$$x^2 + 2x - 15 = 0 \quad (5)$$

$$(x - 3)(x + 5) = 0 \quad (6)$$

This results in a domain of all real numbers but we must exclude $x = 3$ and $x = -5$. In interval notation: $(-\infty, -5) \cup (-5, 3) \cup (3, \infty)$.

(b) $g(x) = \frac{x}{\sqrt{2x^2 - 4x}}$

Solution:

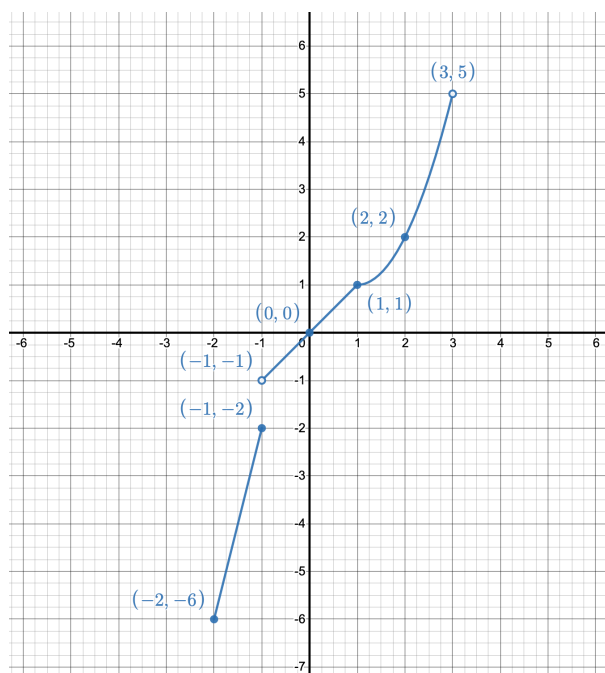
The domain is found when: $2x^2 - 4x > 0$. Factoring we get: $2x(x - 2) > 0$. $2x(x - 2) = 0$ when $x = 0$ and $x = 2$. Setting up a number line and testing values we get the domain: $(-\infty, 0) \cup (2, \infty)$.

(c) $h(x) = 2x^3 - 4x^2 + 6$

Solution:

Since $h(x)$ is a polynomial the domain is $(-\infty, \infty)$.

3. Answer the following for the given graph of $f(x)$. (14 pts)



(a) Find the domain of $f(x)$. Express your answer in interval notation.

Solution:

$$[-2, 3)$$

(b) Find the range of $f(x)$. Express your answer in interval notation.

Solution:

$$[-6, -2] \cup (-1, 5)$$

(c) Find $f(0)$.

Solution:

$$f(0) = 0.$$

(d) Find $(f + f)(2)$.

Solution:

$$(f + f)(2) = f(2) + f(2) = 2 + 2 = 4.$$

(e) Find $(f \circ f)(-1)$.

Solution:

$$(f \circ f)(-1) = f(f(-1)) = f(-2) = -6.$$

(f) $f(x)$ is one-to-one, give a brief sentence to explain why it is one-to-one.

Solution:

$f(x)$ is one-to-one since if $f(x_1) = f(x_2)$ then $x_1 = x_2$. This is commonly referred to as passing the horizontal line test (HLT).

(g) Find the range of $f^{-1}(x)$. Express your answer in interval notation.

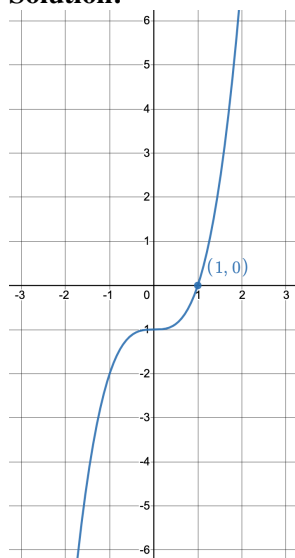
Solution:

$f^{-1}(x)$ exists since $f(x)$ is one-to-one. The range of $f^{-1}(x)$ is the domain of $f(x)$ so the range of $f^{-1}(x)$ is $[-2, 3)$.

4. Sketch the shape of the graph of each of the following on the given set of axes. Label x -intercepts if any: (19 pts)

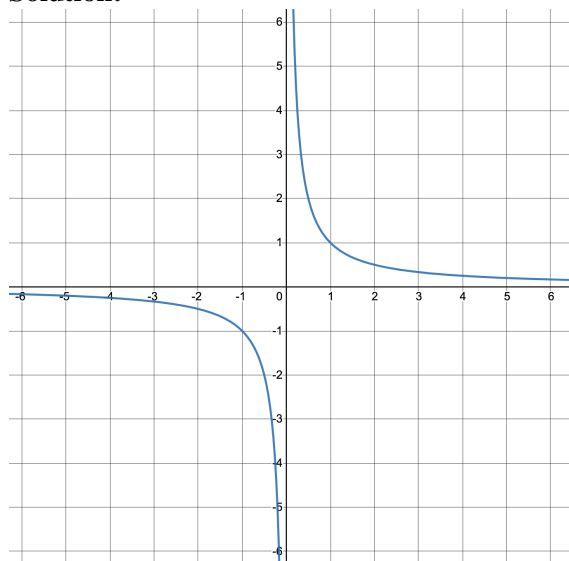
(a) $f(x) = x^3 - 1$

Solution:



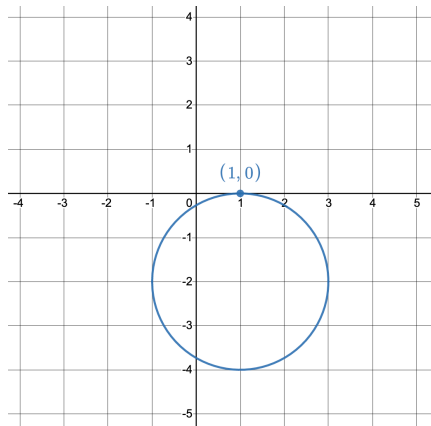
(b) $k(x) = \frac{1}{x}$

Solution:



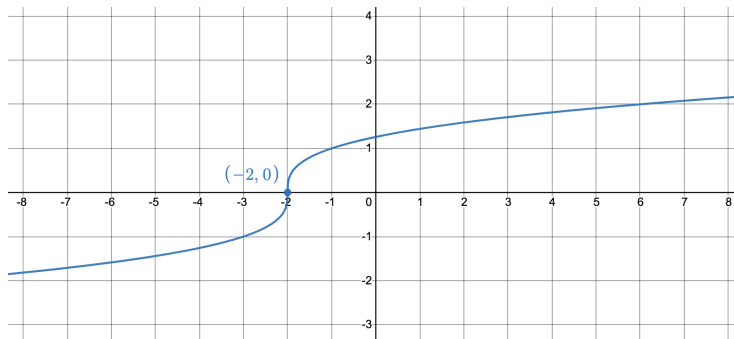
(c) $(x - 1)^2 + (y + 2)^2 = 4$

Solution:



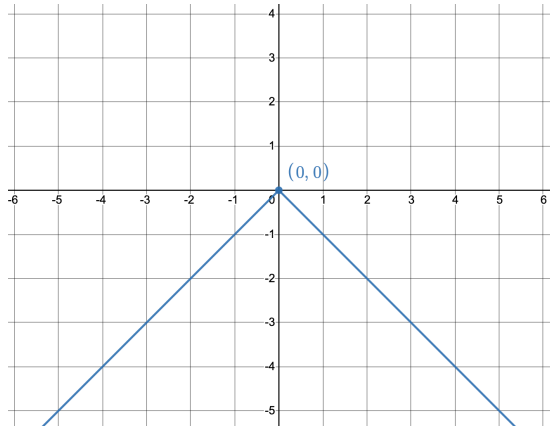
(d) $g(x) = \sqrt[3]{x + 2}$

Solution:



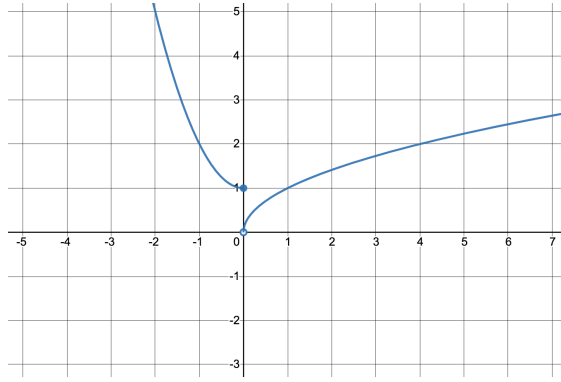
(e) $m(x) = -|x|$

Solution:



$$(f) \quad q(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

Solution:



5. For $g(x) = x^3 - 2$ and $h(x) = \frac{1}{\sqrt[3]{x}}$ answer the following: (16 pts)

(a) Is $g(x)$ odd, even, or neither? Justify your answer to receive credit.

Solution:

Consider $g(-x) = (-x)^3 - 2 = -x^3 - 2$. This does not equal $g(x)$ and does not equal $-g(x) = -(x^3 - 2) = -x^3 + 2$ so this function is **neither** even nor odd. We can further prove $g(x)$ is neither odd nor even by showing $g(1) = 1^3 - 2 = -1$ but $g(-1) = (-1)^3 - 2 = -3$ and since $g(-1) \neq 1$ and $g(-1) \neq -1$ then $g(x)$ is neither odd nor even.

(b) Find $h(\sqrt{x})$ and rewrite your answer with rational exponent(s). Simplify your answer as usual.

Solution:

$$h(\sqrt{x}) = \frac{1}{\sqrt[3]{\sqrt{x}}} = \frac{1}{(x^{1/2})^{1/3}} = \frac{1}{x^{1/6}}.$$

(c) Find $(g \circ h)(x)$ and state its domain in interval notation.

Solution:

$$(g \circ h)(x) = g(h(x)) = g\left(\frac{1}{\sqrt[3]{x}}\right) = \left(\frac{1}{\sqrt[3]{x}}\right)^3 - 2 = \frac{1}{x} - 2. \text{ The domain is } (-\infty, 0) \cup (0, \infty).$$

(d) Find $g^{-1}(x)$ (you may assume the inverse function exists).

Solution:

Replacing $g(x)$ by y and solving for x :

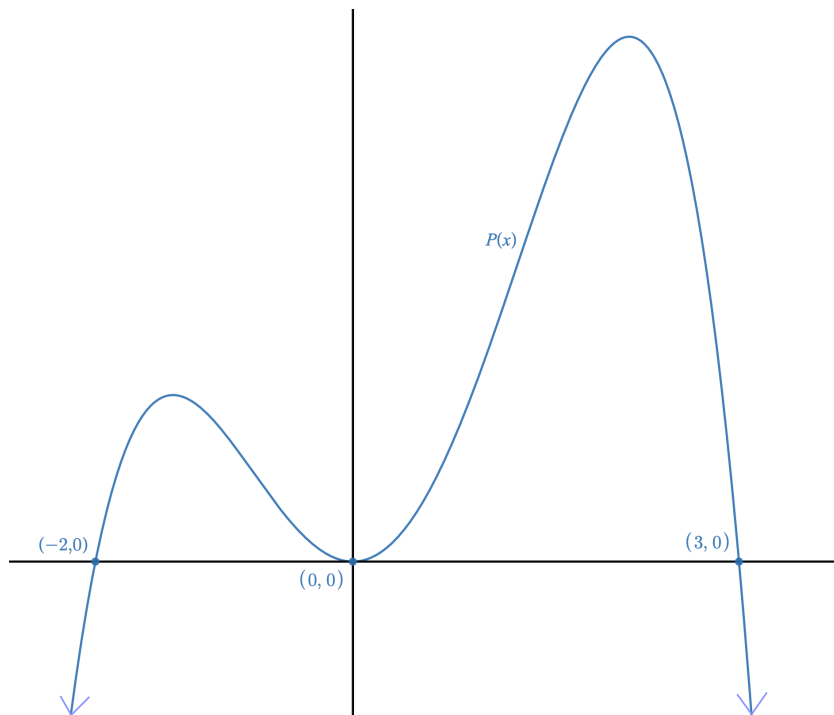
$$y = x^3 - 2 \tag{7}$$

$$y + 2 = x^3 \tag{8}$$

$$\sqrt[3]{y + 2} = x \tag{9}$$

Swapping x and y and replacing y by $g^{-1}(x)$ we get: $g^{-1}(x) = \sqrt[3]{x + 2}$.

6. Consider the given graph of a polynomial $P(x)$ with domain $(-\infty, \infty)$. Write down a degree four polynomial, in factored form, whose graph is consistent with that of $P(x)$: (5 pts)



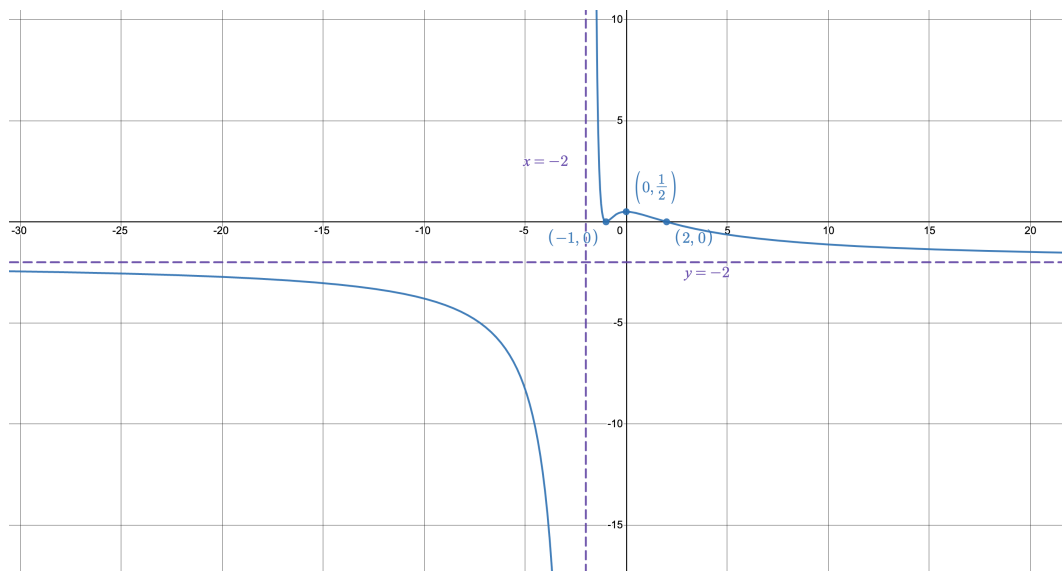
Solution:

$P(x) = -x^2(x + 2)(x - 3)$ satisfies all of the given information.

7. Sketch the shape of the graph of a rational function, $R(x)$, that satisfies **all** of the information. **Label** all intercepts and asymptotes on the graph. (6 pts)

- The graph has a horizontal asymptote: $y = -2$.
- The graph crosses at x -intercept $(2, 0)$.
- The graph bounces (touches but does not cross) at x -intercept $(-1, 0)$.
- The graph has no other x -intercepts and no holes.
- The graph has y -intercept $(0, \frac{1}{2})$.
- The graph has a vertical asymptote: $x = -2$.

Solution:



8. For $R(x) = \frac{3x^3 - 6x^2 - 9x}{x^2 - 9}$ (12 pts)

(a) Find the location (x, y -coordinates) of any hole(s). If there are none state NONE.

Solution:

Factoring we get: $R(x) = \frac{3x^3 - 6x^2 - 9x}{x^2 - 9} = \frac{3x(x^2 - 2x - 3)}{(x - 3)(x + 3)} = \frac{3x(x - 3)(x + 1)}{(x - 3)(x + 3)} = \frac{3x(x + 1)}{x + 3}$
when $x \neq 3$. So we have a hole when $x - 3 = 0$ or $x = 3$.

To find the y -coordinate we plug $x = 3$ into the simplified rational function to get: $\frac{3(3)(3 + 1)}{3 + 3} = 6$.
So the hole is located at $(3, 6)$.

(b) Find any horizontal or slant asymptote(s). If there are none state NONE.

Solution:

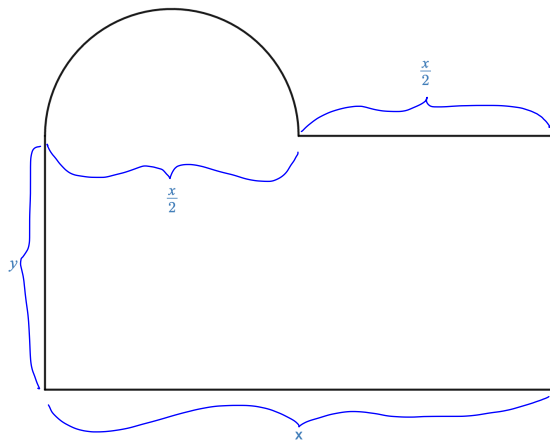
There is a slant asymptote since the degree of the polynomial in the numerator is exactly one larger than the degree of the polynomial in the denominator. By utilizing long division we end up with a slant (oblique) asymptote: $y = 3x - 6$.

(c) Find any vertical asymptote(s). If there are none state NONE.

Solution:

Looking at the reduced rational function $y = \frac{3x(x + 1)}{x + 3}$ there is a vertical asymptote when the denominator is zero. Namely when $x + 3 = 0$ or $x = -3$.

9. The surface of a swimming pool is being designed to have the shape of a semi-circle adjoined to a rectangle with total perimeter 120 ft (see diagram). The diameter of the circle is half the length of one side of the rectangle (as depicted on diagram). The length of one side of the rectangle is y . Write down the area of the pool as a function of x . **You may leave your final answer in an unsimplified form.** (5 pts)



Solution:

We want to find the area as a function of x . The area of the pool is half the area of a circle combined with the area of the rectangle defined in the picture. This is given by: $A = xy + \frac{1}{2}\pi\left(\frac{x}{4}\right)^2$. The perimeter is given by: $P = 120 = x + \frac{x}{2} + 2y + \frac{1}{2}2\pi\frac{x}{4} = \frac{3x}{2} + 2y + \frac{\pi x}{4}$ where $\frac{1}{2}2\pi\frac{x}{4}$ is one half of the circumference of the circle with radius $\frac{x}{4}$. Solving the perimeter equation for y we get: $y = \frac{1}{2}\left(120 - \frac{3x}{2} - \frac{\pi x}{4}\right)$. Plugging this into the area equation we get: $A(x) = \frac{x}{2}\left(120 - \frac{3x}{2} - \frac{\pi x}{4}\right) + \frac{1}{2}\pi\left(\frac{x}{4}\right)^2$. The problem specifies we may leave this answer unsimplified so we do not simplify.